A Globally and Universally Stable Quantity Adjustment Process for an Exchange Economy with Price Rigidities ‡ §

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Abstract

An exchange economy with fixed prices is considered. In this setting a Walrasian equilibrium generically does not exist. In a so-called Drèze equilibrium rationing on excess demand and on excess supply of the non-numeraire commodities is present. In this paper a new type of adjustment process is introduced, namely an adjustment process in quantities or, equivalently, rationing schemes. The total excess demand as a function of the rationing schemes does not satisfy the assumptions under which any of the existing price adjustment processes converges to a Walrasian equilibrium. From the main theorem it follows that the quantity adjustment process converges generically to a Drèze equilibrium. The assumptions made with respect to consumptions sets, preferences and initial endowments are standard. No restrictions are made with respect to the initial state of the economy, which determines the starting point of the adjustment process. Moreover, it follows from the main theorem that the number of Drèze equilibria is generically odd.

1 Introduction

In general equilibrium theory a nice answer to the equilibrium existence problem for an exchange economy with complete markets and a finite number of agents and commodities has been given by the elegant theory as presented in Debreu (1959). The stability of a so-called competitive equilibrium has turned out to be a far more difficult problem. The first price adjustment process was already given in Walras (1874). For a number of special cases the Walrasian tatonnement process in continuous time, as formulated in Samuelson (1941), has been shown to be globally convergent, see for example Arrow and Hurwicz (1958) and Arrow, Block and Hurwicz (1959). However, the counterexamples of Scarf (1960) and the lack of structure of the total excess demand function (see Debreu (1974)) make clear that the Walrasian tatonnement process does not necessarily converge to a competitive equilibrium. For a discrete time version of the Walrasian tatonnement process even chaotic behaviour may be expected (see Day and Pianigiani (1991)). Even if the Walrasian tatonnement process converges to a competitive equilibrium, convergence may take too much time and does not take place, a point of view considered in Blad (1978). Consequently, it is well possible that, at least in the short run, a competitive equilibrium price system is not reached, and therefore trade should take place at a non-competitive equilibrium price system. Clearly, there are several other reasons why trade may take place at a non-competitive equilibrium price system. Even if the competitive equilibrium is stable in an economy, then government intervention, for instance minimum wages or price indexation, might result in a non-competitive price at which trade will take place. Similarly, as the results in Madden (1983), Silvestre (1988), and Bénassy (1993) indicate, a non-Walrasian fixed price equilibrium may result as the outcome of a game played between consumers and firms or the outcome of a game played between workers and shareholders.

When trade takes place at a non-competitive equilibrium price system, several equilibrium concepts are available, being described in Bénassy (1975), Drèze (1975), Younès (1975), van der Laan (1980), and Nguyen and Whalley (1986, 1990). Although existence of each such a non-Walrasian equilibrium concept is shown, again the question of stability for these non-Walrasian equilibria should be addressed. Only Bénassy (1975) accompagnies his equilibrium concept by a dynamic process in discrete time, specifying the amounts the consumers can supply or demand on the various markets at each point in time. However, he does not seem to be concerned with the convergence of such a process.

Many authors consider models where at each moment in time a non-Walrasian equilibrium results. A price adjustment process in continuous time in a world with three commodities is considered by Veendorp (1975) (see also the comment of Laroque (1981)). In this model prices are adjusted on the basis of the effective demands corresponding with a non-Walrasian equilibrium in each time period. Due to the assumptions made, this non-Walrasian equilibrium is unique and can easily be determined. Under more general assumptions it is not clear how this non-Walrasian equilibrium can be attained. Even if at each point in time a non-Walrasian equilibrium results, it is not clear which equilibrium will realize in case of multiple equilibria. Hence an adjustment process is needed to select an equilibrium in this case. Other authors, like Böhm (1992) and Weddepohl and Yildirim (1993) consider overlapping generation models where a non-Walrasian equilibrium results in each period. Again, assumptions are made such that the non-Walrasian equilibrium can easily be determined. In case of more general assumptions it is again less clear whether a non-Walrasian equilibrium will result, and in case of multiple equilibria which equilibrium prevails in the economy.

In this paper a quantity adjustment process in continuous time is considered for an exchange economy with a given, not necessarily competitive, price system. During the process no trade takes place. Under standard assumptions it is shown that this process converges generically from any initial state to a Drèze equilibrium. Moreover, from the main result it follows that the number of Drèze equilibria is generically odd. This extends a result of Laroque and Polemarchakis (1978) where it is shown that the number of Drèze equilibria is finite. The assumptions made in this paper do not exclude the case where rationing occurs according to some priority system, a case excluded by the assumptions in Laroque and Polemarchakis (1978).

In Section 2 a model of an exchange economy with a fixed price system is given and the equilibrium concept of Drèze (1975) is defined. The total excess demand function does not depend on prices, but on rationing schemes instead. A rationing scheme on a market determines for every consumer the maximal amount which can be supplied and the maximal amount which can be demanded on this market. There is no rationing on the market of the numeraire commodity. An adjustment process in rationing schemes, as introduced in this paper, is therefore equivalent to an adjustment process in quantities. More precisely, adjusting the rationing scheme for a consumer on a market is equivalent to adjusting the amount a consumer is allowed to supply or to demand on that market.

In Section 3 the quantity adjustment process is introduced. An initial state of the economy, in this case a specification of rationing schemes on every market, is assumed to be given. In general this initial state is incompatible with a Drèze equilibrium. A description of an adjustment process in rationing schemes is given, starting with the initially specified rationing schemes. The global features of the adjustment process are as follows. Adjustments of rationing schemes are based on the total excess demand on the markets of the non-numeraire commodities and on the change in the rationing schemes compared to the initial state. In case there is a total excess demand on a market at some point in time, then rationing schemes are adjusted in such a way that, compared to the initial state, rationing on the excess demand of this commodity is strengthened and rationing on excess supply is weakened. So, compared to the initial state, consumers are allowed to demand less or to supply more of this commodity. Similarly, in case there is a total excess supply on a market at some point in time, rationing schemes are adjusted in such a way that, compared to the initial state, rationing on the excess demand of this commodity is weakened and rationing on the excess supply is strengthened. Finally, the rationing schemes satisfy all the requirements imposed on them in a Drèze equilibrium also during the adjustment process. These global features make the adjustment process economically attractive. The quantity adjustment process is illustrated in Section 3 for an economy given in Scarf (1960). This economy is such that the Walrasian tatonnement process in prices does not converge to a Walrasian equilibrium price system, unless the initial state is given by the Walrasian equilibrium price system. The basic features of the quantity adjustment process are related to the price adjustment process as defined in van der Laan and Talman (1987). For extensions and interpretations of this process the reader is referred to van den Elzen (1993). This price adjustment process is shown to converge generically to a Walrasian equilibrium in Herings (1994).

In Section 4 it is shown that for almost every economy the quantity adjustment process as defined in Section 3 converges to a Drèze equilibrium for every given initial state of the economy. In the terminology of Saari and Simon (1978) or Saari (1985), this quantity adjustment process is an effective or globally convergent mechanism. In principle it is possible to adjust the rationing schemes according to the processes as formulated in Samuelson (1941), i.e., the well-known Walrasian tatonnement process, Smale (1976), van der Laan and Talman (1987), or in Kamiya (1990). However, the total excess demand as a function of the rationing schemes will in general not satisfy the requirements needed to guarantee the convergence of these processes. Typically, the total excess demand as a function of the rationing schemes does not satisfy assumptions like gross substitutability, does not have the required boundary behaviour, and is not everywhere differentiable.

2 The Model

For $k \in \mathbb{N}$, let I_k denote the set of integers $\{1, \ldots, k\}$, $\mathbb{R}^k_+$ the non-negative orthant of the k-dimensional Euclidean space $\mathbb{R}^k$, $\mathbb{R}^k_{++}$ the set $\{x \in \mathbb{R}^k \mid x_j > 0,\ \forall j \in I_k\}$, 0^k a k-dimensional vector of zeros, and 1^k a k-dimensional vector of ones. When S is a subset of $\mathbb{R}^k$, then $\mathrm{Int}(S)$ denotes the interior of S in $\mathbb{R}^k$ and $\mathrm{cl}(S)$ the closure of S in $\mathbb{R}^k$. In this section the quantity adjustment process is described for a given exchange economy $\mathcal{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m, (\tilde{l}, \tilde{L}), p)$. There are m consumers indexed $i = 1, \ldots, m$ and $n+1$ commodities, indexed $j = 1, \ldots, n+1$. Each consumer is defined by a consumption set X^i, a utility function $u^i : X^i \to \mathbb{R}$, and a bundle of initial endowments ω^i. The vector

$(\omega^{1\top}, \ldots, \omega^{m\top})^\top$ will be denoted by ω. The price system is assumed to be completely fixed and is given by $p \in \mathbb{R}^{n+1}$. Commodity $n+1$ is considered to be a numeraire commodity, hence the price of commodity $n+1$ equals one. In general it will not hold that p is a Walrasian equilibrium price system, i.e., at p the total excess demand of the consumers is not equal to zero. Hence, when a fixed price system p is given, other equilibrium concepts than the Walrasian one have to be used in order to describe the allocation resulting in the economy. In this paper we will follow the approach of Drèze (1975). In this approach rationing on excess demands and on excess supplies is endogenously determined in order to obtain a situation where the total excess demand of each commodity is zero.

If for some consumer $i \in I_m$ a rationing scheme $(l^i, L^i) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$ is given, then the demand of this consumer is denoted by $d^i(l^i, L^i)$ and is equal to the set of best elements according to u^i in the budget set, denoted by $B^i(l^i, L^i)$,

$$B^i(l^i, L^i) = \{x^i \in X^i \mid \forall j \in I_n,\ l^i_j \leq x^i_j - \omega^i_j \leq L^i_j,\ p \cdot x^i \leq p \cdot \omega^i\}.$$

Notice that there is no rationing on the market of the numeraire commodity. Given a rationing scheme $(l^i, L^i) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$, consumer $i \in I_m$ is said to be constrained on his supply on market $k \in I_n$, or equivalently l^i_k is said to be binding for consumer $i \in I_m$, if $\overline{x}^i \in d^i(\overline{l}^i, L^i)$ and $x^i \in d^i(l^i, L^i)$ implies $u^i(\overline{x}^i) > u^i(x^i)$, where $\overline{l}^i$ is the rationing scheme with $\overline{l}^i_j = l^i_j$, $\forall j \in I_n \setminus \{k\}$, and $\overline{l}^i_k = l^i_k - \varepsilon$ for an arbitrary positive real number ε. Under strict quasi-concavity of preferences, an assumption made further on, it is not difficult to show that if a consumer $i \in I_m$ is constrained on his supply on market k, then $x^i \in d^i(l^i, L^i)$ implies $l^i_k = x^i_k - \omega^i_k$, and if consumer $i \in I_m$ is not constrained on his supply on market k, then $x^i \in d^i(l^i, L^i)$ implies $x^i \in d^i(\overline{l}^i, L^i)$. Similar remarks can be made with respect to demand rationing.

The rationing system $(\tilde{l}, \tilde{L})$ specifies all feasible rationing schemes. The function $\tilde{l} : \mathbb{R}^n_+ \to -\mathbb{R}^{mn}_+$ ($\tilde{L} : \mathbb{R}^n_+ \to \mathbb{R}^{mn}_+$) determines the feasible rationing schemes on excess supplies (excess demands). Component $n(i-1)+j$ of $\tilde{l}$ ($\tilde{L}$) is denoted by $\tilde{l}^i_j$ ($\tilde{L}^i_j$). The set of feasible rationing schemes on excess supplies (excess demands) is given by elements of the set $\tilde{l}(\mathbb{R}^n_+)$ ($\tilde{L}(\mathbb{R}^n_+)$). An element $q^1 \in \mathbb{R}^n_+$ determines the rationing vector $\tilde{l}(q^1) = (\tilde{l}^1(q^1), \ldots, \tilde{l}^m(q^1))$ on the supply side with $\tilde{l}^i_j(q^1)$ determining the supply rationing for consumer $i \in I_m$ on market $j \in I_n$. Similarly, an element $q^2 \in \mathbb{R}^n_+$ determines the rationing vector $\tilde{L}(q^2) = (\tilde{L}^1(q^2), \ldots, \tilde{L}^m(q^2))$ on the supply side with $\tilde{L}^i_j(q^2)$ determining the supply rationing for consumer $i \in I_m$ on market $j \in I_n$.

This way of modelling the rationing system allows for many possibilities, like the uniform rationing system, rationing determined by priority, rationing determined by market share, or no restriction at all. For an extensive discussion of this kind of modelling the rationing system it is referred to Laroque and Polemarchakis (1978) and Herings (1992). For example, the uniform rationing system introduced in Drèze (1975) is defined by setting for every $i \in I_m$, $\tilde{l}^i(q^1) = -q^1$, for all $q^1 \in \mathbb{R}^n_+$, and $\tilde{L}^i(q^2) = q^2$, for all $q^2 \in \mathbb{R}^n_+$. A Drèze equilibrium is defined as follows.

Definition 2.1

A Drèze equilibrium of the economy $\mathcal{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m, (\tilde{l}, \tilde{L}), p)$ is an element

$$(x^{*1}, \ldots, x^{*m}, l^{*1}, \ldots, l^{*m}, L^{*1}, \ldots, L^{*m}) \in \prod_{i=1}^{m} X^i \times -\mathbb{R}^{mn}_+ \times \mathbb{R}^{mn}_+$$

satisfying

1. *$\forall i \in I_m$, x^{*i} is a best element for u^i in $B^i(l^{*i}, L^{*i})$;*
2. *$\sum_{i=1}^m (x^{*i} - \omega^i) = 0^{n+1}$;*
3. *$\forall j \in I_n$, $x_j^{*h} - \omega_j^h = l_j^{*h}$ for some $h \in I_m$ implies $x_j^{*i} - \omega_j^i < L_j^{*i}$, $\forall i \in I_m$, and $x_j^{*h} - \omega_j^h = L_j^{*h}$ for some $h \in I_m$ implies $x_j^{*i} - \omega_j^i > l_j^{*i}$, $\forall i \in I_m$;*
4. *$(l^{*1}, \ldots, l^{*m}) \in \tilde{l}(\mathbb{R}^n_+)$ and $(L^{*1}, \ldots, L^{*m}) \in \tilde{L}(\mathbb{R}^n_+)$.*

In a Drèze equilibrium every consumer behaves optimally given his utility function, consumption set, initial endowments, the price system, and the equilibrium rationing schemes. The second requirement states that all markets clear. The third condition expresses that markets are transparent. Under the assumption of strictly quasi-concave utility functions this condition implies that at most one side of a market experiences binding rationing. Finally, in the last condition it is required that the rationing schemes are feasible.

For the remainder of this section, let an economy $\mathcal{E}$ satisfying the following assumptions be given.

A1. For every $i \in I_m$, X^i is a convex, closed, non-empty subset of $\mathbb{R}^{n+1}_+$ and $X^i + \mathbb{R}^{n+1}_+ \subset X^i$.

A2. For every $i \in I_m$, the utility function $u^i : X^i \to \mathbb{R}$ is continuous, strictly quasi-concave, and strictly increasing in the amount of the numeraire commodity.

A3. For every $i \in I_m$, the initial endowments ω^i are an element of $\mathrm{Int}(X^i)$.

A4. The price system p is an element of $\mathbb{R}^n_{++} \times \{1\}$.

A5. The functions $\tilde{l} : \mathbb{R}^n_+ \to -\mathbb{R}^{mn}_+$ and $\tilde{L} : \mathbb{R}^n_+ \to \mathbb{R}^{mn}_+$ specifying the rationing system are continuous on $\mathbb{R}^n_+$ and satisfy for every $i \in I_m$, $j \in I_n$, q^1, $\tilde{q}^1$, q^2, $\tilde{q}^2 \in \mathbb{R}^n_+$

$$\begin{array}{ll} \tilde{l}^i_j(q^1) = \tilde{l}^i_j(\tilde{q}^1) \text{ if } q^1_j = \tilde{q}^1_j, & \tilde{L}^i_j(q^2) = \tilde{L}^i_j(\tilde{q}^2) \text{ if } q^2_j = \tilde{q}^2_j, \\ \tilde{l}^i_j(q^1) = 0 \text{ if } q^1_j = 0, & \tilde{L}^i_j(q^2) = 0 \text{ if } q^2_j = 0, \\ \tilde{l}^i_j(q^1) \to -\infty \text{ if } q^1_j \to \infty, & \tilde{L}^i_j(q^2) \to \infty \text{ if } q^2_j \to \infty. \end{array}$$

Assumption A5 guarantees that rationing schemes are sufficiently flexible, while the rationing on a market $j \in I_n$ only depends on q_j. Both the case with complete rationing on the demand or the supply of market j, and the case without rationing on the demand or the supply of market j are possible. Although this is not assumed, the function $\tilde{l}$ is considered to be decreasing and $\tilde{L}$ is considered to be increasing, whenever an intuitive explanation or interpretation of the model is given. Under Assumptions A1-A5 on the economy $\mathcal{E}$ the existence of a Drèze equilibrium can be shown as in Drèze (1975). Furthermore, it is not difficult to show that

$$\forall j \in I_n,\ \exists \underline{q}_j \text{ such that } \forall i \in I_m,\ \tilde{l}^i_j(q^1) \text{ is not binding if } q^1_j \geq \underline{q}_j, \tag{1}$$

$$\forall j \in I_n,\ \exists \overline{q}_j \text{ such that } \forall i \in I_m,\ \tilde{L}^i_j(q^2) \text{ is not binding if } q^2_j \geq \overline{q}_j. \tag{2}$$

For example, it follows immediately that if $\underline{q}_j$ is chosen such that $\forall q^1_j \geq \underline{q}_j$, $\tilde{l}^i_j(q^1) < -\omega^i_j$, $\forall i \in I_m$, then the first condition is satisfied and, similarly, if $\overline{q}_j$ is chosen such that $\forall q^2_j \geq \overline{q}_j$, $\tilde{L}^i_j(q^2) > \frac{p \cdot \omega^i}{p_j} - \omega^i_j$, $\forall i \in I_m$, then the second condition is satisfied. Now define the cube $Q^n = \prod_{j=1}^n [0,1]$ and the function $(\hat{l}, \hat{L}) : Q^n \rightarrow -\mathbb{R}^{mn}_+ \times \mathbb{R}^{mn}_+$ by

$$(\hat{l}(q), \hat{L}(q)) = (\tilde{l}(2\underline{q}_1 q_1, \ldots, 2\underline{q}_n q_n), \tilde{L}(2\overline{q}_1(1-q_1), \ldots, 2\overline{q}_n(1-q_n))),\ \forall q \in Q^n.$$

It is easily verified that demand rationing is not binding on market $j \in I_n$ if $q_j \leq \frac{1}{2}$ and supply rationing is not binding on market $j \in I_n$ if $q_j \geq \frac{1}{2}$. So the rationing scheme $(\hat{l}(q), \hat{L}(q))$ is not binding for any consumer on market j if $q_j = \frac{1}{2}$. When q_j increases from 0 to $\frac{1}{2}$, then supply rationing changes from being equal to zero, i.e., no supply is possible of commodity j, to being non-binding for any consumer. When $q_j \in [\frac{1}{2}, 1]$, then supply rationing remains non-binding for every consumer. When $q_j \in [0, \frac{1}{2}]$, then demand rationing is non-binding for every consumer and when q_j increases from $\frac{1}{2}$ to 1 then demand rationing changes from being non-binding to being completely binding. Hence, if q_j is increased then the demand of commodity j has a tendency to fall. In this sense the effect on the economy of an increment of q_j resembles the effect of an increment of the price of commodity j.

Let a rationing scheme $(l^i, L^i) \in -\mathbb{R}^n_+ \times \mathbb{R}^n_+$ be given. By the strict quasi-concavity of u^i it is clear that $d^i(l^i, L^i)$ is uniquely defined. From the results in Drèze (1975) it follows immediately that d^i is a continuous function defined on $-\mathbb{R}^n_+ \times \mathbb{R}^n_+$ and mapping into X^i. Now, for every $i \in I_m$, the individual demand function $\hat{d}^i : Q^n \rightarrow \mathbb{R}^{n+1}$ and the total excess demand function $\hat{z} : Q^n \rightarrow \mathbb{R}^{n+1}$ of the economy $\mathcal{E}$ are defined by

$$\begin{aligned} \hat{d}^i(q) &= d^i(\hat{l}^i(q), \hat{L}^i(q)),\ \forall q \in Q^n, \\ \hat{z}(q) &= \sum_{i=1}^m (\hat{d}^i(q) - \omega^i),\ \forall q \in Q^n. \end{aligned}$$

The following theorem is easily shown, using the continuity of d^i, $\forall i \in I_m$.

Theorem 2.2

Let the economy $\mathcal{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m, (\widetilde{l}, \widetilde{L}), p)$ satisfy Assumptions A1-A5. Then the total excess demand function $\widehat{z} : Q^n \to \mathbb{R}^{n+1}$ satisfies the following conditions:

1. *$\widehat{z}$ is continuous on Q^n;*

2. $\forall q \in Q^n,\ \forall j \in I_n,\ \widehat{z}_j(q) \geq 0$ *if* $q_j = 0$;

3. $\forall q \in Q^n,\ \forall j \in I_n,\ \widehat{z}_j(q) \leq 0$ *if* $q_j = 1$;

4. $\forall q \in Q^n,\ p \cdot \widehat{z}(q) = 0$;

5. $\widehat{z}(q^*) = 0^{n+1}$ *iff* $(\widehat{d}^1(q^*), \ldots, \widehat{d}^m(q^*), \widehat{l}(q^*), \widehat{L}(q^*))$ *is a Drèze equilibrium of the economy* $\mathcal{E}$.

It is not difficult to show that all the different Drèze equilibria of the economy $\mathcal{E}$ are obtained by considering all the zero points of $\widehat{z}$. Here two Drèze equilibria are considered to be different if the equilibrium allocations are different or if the binding rationing schemes are different. Since a point $q^* \in Q^n$ satisfying $\widehat{z}(q^*) = 0^{n+1}$ induces in a unique way a Drèze equilibrium $(\widehat{d}^1(q^*), \ldots, \widehat{d}^m(q^*), \widehat{l}(q^*), \widehat{L}(q^*))$ such a point q^* will also be called a Drèze equilibrium in the sequel.

3 The Quantity Adjustment Process

In this section a quantity adjustment process will be defined for the model of the previous section. In the model described in Section 2 the state of the markets is completely determined by the prevailing rationing schemes (l, L). Since all feasible rationing schemes are obtained by the function $(\widehat{l}, \widehat{L}) : Q^n \to -\mathbb{R}^{mn} \times \mathbb{R}^{mn}$, it is possible to describe the state of an economy by an element $q \in Q^n$, inducing the rationing schemes $(\widehat{l}(q), \widehat{L}(q))$. The basic idea of the quantity adjustment process will be to increase (decrease) q_j in the case of positive (negative) excess demand on a market $j \in I_n$.

Let $v \in Q^n$ denote the initial state of the economy. The initial state $v = \frac{1}{2}1^n$ is interesting from an economic point of view, since at this state no consumer is rationed on any market. In this case, initially, each consumer expresses his notional demand for every commodity, given the price system p. Subsequently, demand will be rationed on markets with a positive excess demand, and supply will be rationed on markets with a negative excess demand. Other interesting initial states are $v = 0^n$ and $v = 1^n$ which correspond to situations with complete rationing of supplies and complete rationing of demands, respectively. For example, in the case of complete rationing of supplies all consumers are restricted to express a non-negative excess demand of a commodity. In general, this will lead to positive excess demands on all markets (except on the market of the numeraire commodity). Then these positive excess demands are allocated to potential suppliers by allowing some supply on all markets. This will change the excess demand on a particular market for two reasons. First, there is a direct effect, since supply of a commodity will directly decrease the positive excess demand of this commodity. Secondly, there is an indirect spill-over effect, since the opportunity of supply on other markets will change the demand on a market. The sign of this spill-over effect is ambiguous. However, other initial states are interesting too. When a sequence of temporary equilibria is considered, then the rationing schemes of the previous period determine the initial state for the current period. Since there are many possible initial states of an economy, v is allowed to be an arbitrary point in Q^n in this paper.

Given some initial state $v \in Q^n$, v_j is the maximal decrease of q_j possible on market j, and $1 - v_j$ is the maximal increase of q_j possible on market j. Let $q \in Q^n$ be a point reached by the quantity adjustment process. The quantity adjustment process is such that if there is an excess demand on a market $j \in I_n$, then q_j has been increased maximally over all commodities towards one, and if there is an excess supply on a market $j \in I_n$, then q_j has been decreased maximally over all commodities towards zero. More precisely, if an element $q \in Q^n$ has been reached by the quantity adjustment process, then there exists a number $\mu \in [0,1]$ such that for every $j \in I_n$

$$\begin{aligned} q_j &= \mu v_j \text{ if } \hat{z}_j(q) < 0, \\ \mu v_j \leq q_j &\leq 1 - \mu + \mu v_j \text{ if } \hat{z}_j(q) = 0, \\ q_j &= 1 - \mu + \mu v_j \text{ if } \hat{z}_j(q) > 0. \end{aligned} \tag{3}$$

Clearly, the behaviour of the quantity adjustment process depends heavily on the state of the various markets, i.e., whether there is a positive excess demand on a market, or a market is in equilibrium, or there is a negative excess demand on a market. The state of the markets will be described by a sign vector $s \in \mathbb{R}^n$, so for every $j \in I_n$, $s_j \in \{-1, 0, +1\}$. Define the set of feasible sign vectors $\mathcal{S}$ by

$$\begin{aligned} \mathcal{S} \;=\; \{s \in \mathbb{R}^n \mid & \forall j \in I_n,\ s_j \in \{-1,0,+1\},\ \exists j' \in I_n \text{ such that } s_{j'} \neq 0, \\ & \forall j \in I_n, v_j = 0 \text{ implies } s_j \neq -1 \text{ and } v_j = 1 \text{ implies } s_j \neq +1\}. \end{aligned}$$

Notice that $\mathcal{S} \neq \emptyset$. For every $s \in \mathcal{S}$, define the sets $I^-(s) = \{j \in I_n \mid s_j = -1\}$, $I^0(s) = \{j \in I_n \mid s_j = 0\}$, and $I^+(s) = \{j \in I_n \mid s_j = +1\}$. These sets will denote the markets for which there is excess supply, equilibrium, and excess demand, respectively. Moreover, let $i^-(s)$, $i^0(s)$, and $i^+(s)$ denote the number of elements in the sets $I^-(s)$, $I^0(s)$, and $I^+(s)$, respectively. Define the sets $A(s)$, $B(s)$, and $C(s)$ by

$$A(s) \;=\; \{q \in Q^n \mid \exists \mu \in [0,1] \text{ such that } \forall j \in I^-(s),\ q_j = \mu v_j,$$

$$\begin{aligned}
&\forall j \in I^0(s),\ \mu v_j \le q_j \le 1-\mu+\mu v_j,\\
&\forall j \in I^+(s),\ q_j = 1-\mu+\mu v_j\},
\end{aligned}$$

$$\begin{aligned}
B(s) &= \{q \in Q^n \mid \forall j \in I^-(s),\ \hat{z}_j(q) \le 0,\ \forall j \in I^0(s),\ \hat{z}_j(q)=0,\ \text{and}\ \forall j \in I^+(s),\ \hat{z}_j(q) \ge 0\},\\
C(s) &= A(s) \cap B(s).
\end{aligned}$$

Hence, when $q \in C(s)$ for some $s \in \mathcal{S}$, then $s_j = -1$ ($s_j = +1$) for some $j \in I_n$ implies that there is a non-positive (non-negative) excess demand on market j, while the value of q_j is minimal (maximal) relative to v_j, i.e., relative to the situation at the starting point demand rationing on market j has been weakened (tightened) or supply rationing on market j has been tightened (weakened). Every state $q \in Q^n$ reached by the quantity adjustment process will shown to be an element of the set C defined by

$$C = \cup_{s\in\mathcal{S}} C(s).$$

Notice that for every $s \in \mathcal{S}$, the set $B(s)$ does not depend on the initial state v. The regions in Q^n determined by the sets $A(s)$, $s \in \mathcal{S}$, however, do depend on v. The number of different regions $A(s)$ equals the number of feasible sign vectors and also depends on v. This is illustrated for the case $n = 2$ in the Figures I, II, III, and IV. In order to obtain all states $q \in Q^n$ with the properties given in (3) it is sufficient to restrict attention to the points $q \in C$, so only points in the set $C(s)$ corresponding with a feasible sign vector s. If, for instance, $v_j = 0$ for some $j \in I_n$, then the quantity adjustment process cannot reach a state $\tilde{q}$ that induces a negative excess demand on market j. According to (3) such a state must satisfy $\tilde{q}_j = 0$, implying a non-negative excess demand for commodity j by Theorem 2.2.2.

In the next theorem it is shown that $v \in C$ and for every $q^* \in Q^n$ satisfying $\hat{z}(q^*) = 0^{n+1}$ it holds that $q^* \in C$, i.e., $q^* \in C$ for every Drèze equilibrium q^*.

Theorem 3.1

Let the economy $\mathcal{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m, (\tilde{l}, \tilde{L}), p)$ satisfying Assumptions A1-A5 and the initial state $v \in Q^n$ be given. Then $v \in C$ and $q^* \in C$ for every Drèze equilibrium $q^* \in Q^n$.

Proof

Consider the total excess demand $\hat{z}(v)$ at the starting point v. If $\hat{z}(v) = 0^{n+1}$, then, for every s in the non-empty set $\mathcal{S}$, $v \in C(s)$, so $v \in C$. If $\hat{z}(v) \neq 0^{n+1}$, then $\hat{z}_j(v) \neq 0$ for some $j \in I_n$ by Theorem 2.2.4. For every $j \in I_n$, define $\bar{s}_j = -1$ if $\hat{z}_j(v) < 0$, $\bar{s}_j = 0$ if $\hat{z}_j(v) = 0$, and $\bar{s}_j = +1$ if $\hat{z}_j(v) > 0$. Then $\bar{s}$ is a feasible sign vector by Theorems 2.2.2 and 2.2.3. Clearly, $v \in C(\bar{s}) \subset C$.

Let $q^* \in Q^n$ be a Drèze equilibrium. Define $\bar{\mu}$ as the maximal value of μ such that for every $j \in I_n$, $q_j^* \ge \mu v_j$ and $1 - q_j^* \ge \mu(1-v_j)$. It is easily verified that $0 \le \bar{\mu} \le 1$. When for some $j' \in I_n$ it holds that both $q_{j'}^* = \bar{\mu} v_{j'}$ and $1 - q_{j'}^* = \bar{\mu}(1 - v_{j'})$, then it follows that $\bar{\mu} = 1$ and

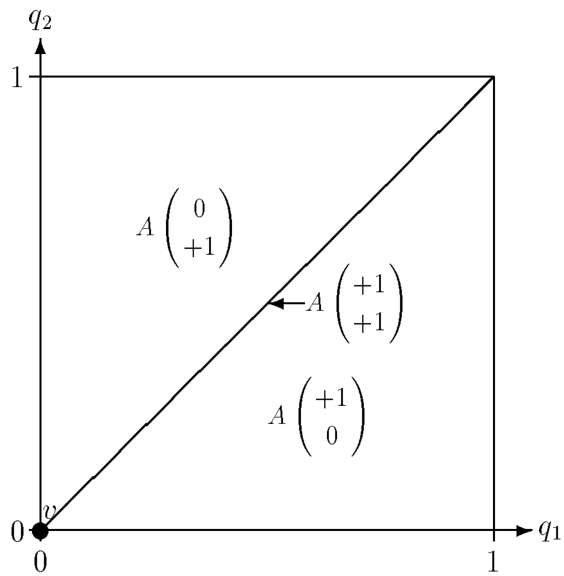

Figure I

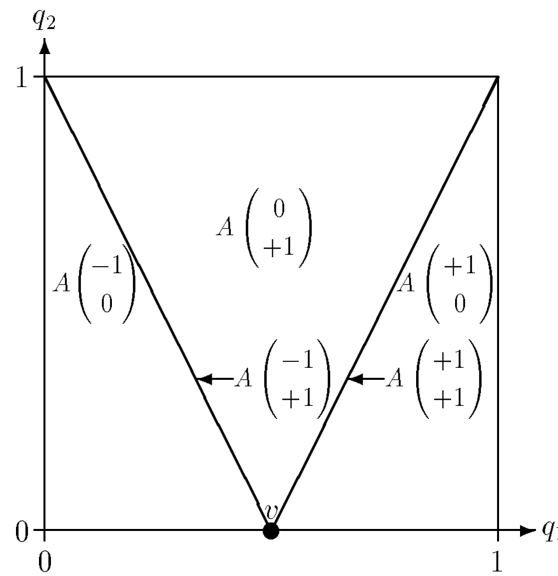

Figure II

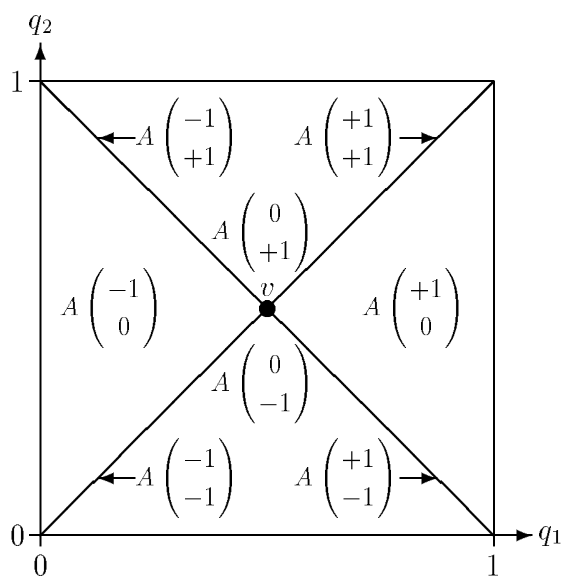

Figure III

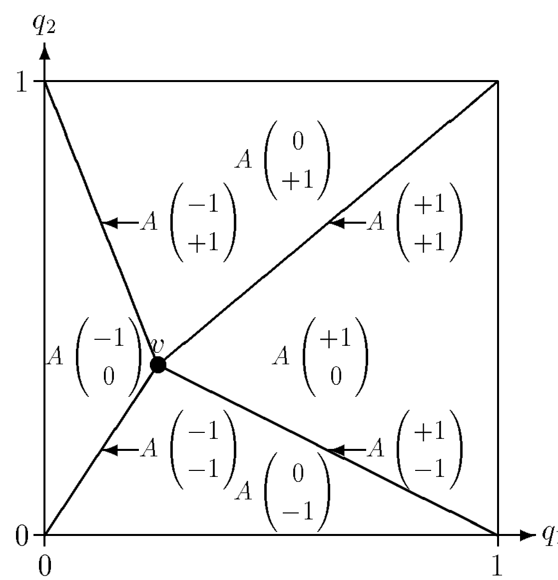

Figure IV

therefore that $q_j^* = v_j$, $\forall j \in I_n$. It follows from above that $q^* \in C$ in this case. When, for every $j \in I_n$, $q_j^* > \overline{\mu} v_j$ or $1 - q_j^* > \overline{\mu}(1 - v_j)$, then define $\overline{s} \in \mathbb{R}^n$ by $\overline{s}_j = -1$ if $q_j^* = \overline{\mu} v_j$ and $v_j > 0$, $\overline{s}_j = +1$ if $1 - q_j^* = \overline{\mu}(1 - v_j)$ and $v_j < 1$, and $\overline{s}_j = 0$, otherwise. By the definition of $\overline{\mu}$ it holds that $q_{j'}^* = \overline{\mu} v_{j'}$ for some $j' \in I_n$ with $v_{j'} > 0$, or $1 - q_{j'}^* = \overline{\mu}(1 - v_{j'})$ for some $j' \in I_n$ with $v_{j'} < 1$. Hence $\overline{s} \in \mathcal{S}$. It follows immediately that $q^* \in C(\overline{s}) \subset C$. Q.E.D.

After defining some topological concepts it is now possible to give a precise definition of the quantity adjustment process. A subset S of a topological space X is connected if it is not the union of two non-empty, disjoint sets, which are open in the induced topology on S. The component of a point x in a subset S of a topological space X is the union of all connected subsets of S containing x. It is not difficult to show that each component is connected and therefore the component of an element x is the largest connected subset of S containing x. The approach chosen to describe the quantity adjustment process is closely related to the approach chosen in Smale (1976, 1981), van der Laan and Talman (1987), and Kamiya (1990) to describe price adjustment processes. The quantity adjustment process is defined as a set of points in Q^n, containing the starting point v. It will be shown, under suitable differentiability conditions, that this set is generically homeomorphic to the closed unit interval having the starting price system v and a point $q^* \in Q^n$ satisfying $\widehat{z}(q^*) = 0^{n+1}$ as boundary points. Moreover, under these assumptions the quantity adjustment process will be shown to be generically a 1-dimensional piecewise twice continuously differentiable manifold and can be described by a system of differential equations, see for example Garcia and Zangwill (1981).

Definition 3.2
Let the economy $\mathcal{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m, (\widetilde{l}, \widetilde{L}), p)$ satisfying Assumptions A1-A5 and the initial state $v \in Q^n$ be given. Then the quantity adjustment process is the component of the set C containing the starting point v.

A subset T of $\mathbb{R}^k$ is an arc if it is homeomorphic to the unit interval $[0, 1]$. A subset T of $\mathbb{R}^k$ is a loop if it is homeomorphic to the unit circle, i.e., the set $\{x \in \mathbb{R}^2 \mid (x_1)^2 + (x_2)^2 = 1\}$.

Definition 3.3
Let the economy $\mathcal{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m, (\widetilde{l}, \widetilde{L}), p)$ satisfying Assumptions A1-A5 and the initial state $v \in Q^n$ be given. The quantity adjustment process is convergent if either $\widehat{z}(v) = 0^{n+1}$, or $\widehat{z}(v) \neq 0^{n+1}$ and the component of the set C containing v is an arc having v and a Drèze equilibrium q^* of the economy $\mathcal{E}$ as its boundary points.

If the quantity adjustment process is convergent, then there exists a continuous function $\pi : [0, 1] \to C$ satisfying $\pi(0) = v$ and $\pi(1) = q^*$, where q^* is a Drèze equilibrium. The continuous function π describes the explicit time path of the process. In the case the adjustment process is described by a system of differential equations, this path corresponds to its trajectory.

The quantity adjustment process has a natural economic interpretation and can be described as follows. First the sign of the total excess demand on the non-numeraire markets is evaluated at the initial state v. Consider first an initial state in the interior of Q^n. In Section 4 it will be shown that then generically $\hat{z}_j(v) \neq 0$, $\forall j \in I_n$. Define the feasible sign vector $s^0 \in \mathcal{S}$ by $s_j^0 = +1$ if $\hat{z}_j(v) > 0$ and $s_j^0 = -1$ if $\hat{z}_j(v) < 0$. Now the process starts by leaving v along the ray $A(s^0)$. Thus, for those $j \in I_n$ with $\hat{z}_j(v) < 0$, q_j will be proportionally decreased and when $\hat{z}_j(v) > 0$, then q_j will be proportionally increased. Hence, rationing on excess supply is weakened or rationing on excess demand is strengthened for those commodities $j \in I_n$ with $\hat{z}_j(v) > 0$, and the other way around for those $j \in I_n$ with $\hat{z}_j(v) < 0$. This will tend to decrease the absolute value of the total excess demand on every market. Rationing schemes are adjusted in this way until one of the markets, say market j', attains an equilibrium situation. In Section 4 it will be shown that, generically, either this happens for exactly one market $j' \in I_n$ at a state $\bar{q}$ with $0 < \bar{q}_{j'} < 1$, or the boundary of Q^n is reached at a point q^* satisfying $\hat{z}(q^*) = 0^{n+1}$. Consider the first case. Then the process continues by keeping market j' in equilibrium, while $q_{j'}$ is relatively increased (decreased) if there was a negative (positive) excess demand on market j' before attaining equilibrium. For the other markets $j \in I_n \setminus \{j'\}$, q_j is decreased maximally if $\hat{z}_j(\bar{q}) < 0$, and q_j is increased further maximally if $\hat{z}_j(\bar{q}) > 0$. So a path in $C(s^1)$ is followed, where $s_{j'}^1 = 0$, and $s_j^1 = s_j^0$, $\forall j \in I_n \setminus \{j'\}$.

The general case is as follows. Assume the process follows a path in $C(s^N)$, for some $N \in \mathbb{N}$. In Section 4 it will be shown that generically the following cases may result. Either a point q^* is reached satisfying $z(q^*) = 0^{n+1}$, i.e., a Drèze equilibrium is obtained. Or some market $j' \in I^-(s^N)$ attains an equilibrium situation, in which case a path in $C(s^{N+1})$ is followed, where s^{N+1} is a feasible sign vector defined by $s_{j'}^{N+1} = 0$, and $s_j^{N+1} = s_j^N$, $\forall j \in I_n \setminus \{j'\}$. Or $q_{j'}$, for some $j' \in I^0(s^N)$, $(1 - q_{j'}$, for some $j' \in I^0(s^N))$ has become relatively minimal, in which case a path in $C(s^{N+1})$ is followed, where s^{N+1} is a feasible sign vector defined by $s_{j'}^{N+1} = -1$ $(s_{j'}^{N+1} = +1)$ and $s_j^{N+1} = s_j^N$, $\forall j \in I_{n+1} \setminus \{j'\}$. It will be shown in Section 4 that generically $C(s)$ is a finite collection of arcs and loops for every $s \in \mathcal{S}$ and that the process will reach a Drèze equilibrium after generating a finite number of sign vectors.

In the case where v lies in the boundary of Q^n it is not a degenerate case that a market $j \in I_n$ for which $v_j = 0$ or $v_j = 1$ is in equilibrium. For example consider the case $v = 0^n$. If p_j is sufficiently high for every $j \in I_n$, then every consumer wants to supply every commodity j in exchange for the numeraire commodity. Hence v is a Drèze equilibrium, at which all supply is completely rationed. In the cases where v lies in the boundary of Q^n it may happen that the process starts with keeping some markets in equilibrium. The interpretation of the process remains the same.

The adjustment process will now be illustrated for the first example given in Scarf (1960). Scarf showed that the Walrasian tatonnement process as formulated by Samuelson (1941) does not converge to the Walrasian equilibrium price system in this example. There are three consumers, three commodities, $X^i = \mathbb{R}^3_+$, $u^i(x^i_1, x^i_2, x^i_3) = \min\{x^i_i, x^i_{i+1}\}$, $\forall x^i \in X^i$, and $\omega^i = e^i$, $\forall i \in I_3$, where $i+1$ is defined as 1 when $i = 3$ and e^i denotes the i-th unit vector in $\mathbb{R}^3$. In this paper the example is extended by choosing a fixed price system $p = (3, 2, 1)$ and considering uniform rationing schemes. To avoid working with demand correspondences instead of demand functions, the following modification with respect to the preferences of the consumers is made. This modification makes no difference at all for the Walrasian tatonnement process. For every consumer $i \in I_3$ it holds that $\widehat{x}^i \succeq^i \widetilde{x}^i$ iff $\min\{\widehat{x}^i_i, \widehat{x}^i_{i+1}\} > \min\{\widetilde{x}^i_i, \widetilde{x}^i_{i+1}\}$ or both $\min\{\widehat{x}^i_i, \widehat{x}^i_{i+1}\} = \min\{\widetilde{x}^i_i, \widetilde{x}^i_{i+1}\}$ and $\max\{\widehat{x}^i_i, \widehat{x}^i_{i+1}\} \geq \max\{\widetilde{x}^i_i, \widetilde{x}^i_{i+1}\}$. It is easily verified that supply rationing cannot be binding if $l^i_j \leq -1$ and demand rationing cannot be binding if $L^i_j \geq 1$. Hence the rationing system $(\widehat{l}, \widehat{L}) : Q^2 \to \mathbb{R}^6$ can be defined by $(\widehat{l}(q), \widehat{L}(q)) = (-2q, -2q, -2q, (2,2)^\top - 2q, (2,2)^\top - 2q, (2,2)^\top - 2q)$. It is easily verified that the resulting total excess demand function $\widehat{z} : Q^2 \to \mathbb{R}^2$ is defined by

$$\widehat{z}(q)^\top = \begin{cases} (\frac{1}{4} - 2q_1, 3q_1 - 2q_2, -\frac{3}{4} + 4q_2), & 0 \leq q_1 \leq \frac{1}{5},\ 0 \leq q_2 \leq \frac{1}{6}, \\ (\frac{1}{4} - 2q_1, -\frac{1}{3} + 3q_1, -\frac{1}{12}), & 0 \leq q_1 \leq \frac{1}{5},\ \frac{1}{6} \leq q_2,\ 2 - 3q_1 - 2q_2 \geq 0, \\ (-1\frac{1}{12} + 1\frac{1}{3}q_2, 1\frac{2}{3} - 2q_2, -\frac{1}{12}), & q_1 \leq \frac{7}{8},\ \frac{7}{10} \leq q_2 \leq 1,\ 2 - 3q_1 - 2q_2 \leq 0, \\ (-\frac{3}{20}, \frac{3}{5} - 2q_2, -\frac{3}{4} + 4q_2), & \frac{1}{5} \leq q_1 \leq \frac{7}{8},\ 0 \leq q_2 \leq \frac{1}{6}, \\ (-\frac{3}{20}, \frac{4}{15}, -\frac{1}{12}), & \frac{1}{5} \leq q_1 \leq \frac{7}{8},\ \frac{1}{6} \leq q_2 \leq \frac{7}{10}, \\ (1\frac{3}{5} - 2q_1, \frac{3}{5} - 2q_2, -6 + 6q_1 + 4q_2), & \frac{7}{8} \leq q_1 \leq 1,\ 0 \leq q_2 \leq \frac{1}{6}, \\ (1\frac{3}{5} - 2q_1, \frac{4}{15}, -5\frac{1}{3} + 6q_1), & \frac{7}{8} \leq q_1 \leq 1,\ \frac{1}{6} \leq q_2 \leq \frac{7}{10}, \\ (\frac{2}{3} - 2q_1 + 1\frac{1}{3}q_2, 1\frac{2}{3} - 2q_2, -5\frac{1}{3} + 6q_1), & \frac{7}{8} \leq q_1 \leq 1,\ \frac{7}{10} \leq q_2 \leq 1. \end{cases}$$

In Figure V the sets $B(s)$ corresponding with the example above are shown for every $s \in \mathcal{S}$. The unique Drèze equilibrium is given by $q^* = (\frac{8}{9}, \frac{5}{6})^\top$. There is binding demand rationing on both markets, $L^* = (\frac{2}{9}, \frac{1}{3})^\top$, $x^{*1} = (\frac{7}{9}, \frac{1}{3}, 0)^\top$, $x^{*2} = (0, \frac{2}{3}, \frac{2}{3})^\top$, and $x^{*3} = (\frac{2}{9}, 0, \frac{1}{3})^\top$. Consumer 1 faces binding rationing on the demand of commodity 2, consumer 3 faces binding rationing on the demand of commodity 1, while consumer 2 obtains his most preferred unconstrained consumption bundle. Notice that in the unique Drèze equilibrium there is rationing of the demands on both markets, while the notional total excess demand given $p = (3, 2, 1)^\top$ yields excess supply on market 1 and excess demand on market 2. In Figure VI the adjustment process is depicted if the initial state is given by $v = (\frac{1}{2}, \frac{1}{2})^\top$. In the initial state v all consumers express their notional demand. Since there is a negative excess demand on market 1 and a positive excess demand on market 2, q_1 is decreased and q_2 is increased. The adjustment process follows a path in the set $C((-1, +1)^\top)$. Hence, the rationing on supply on market 1 and the rationing on demand on market 2 becomes

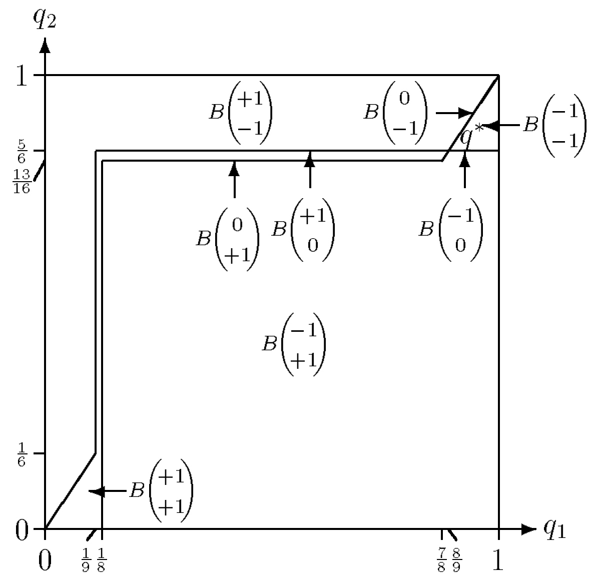

Figure V

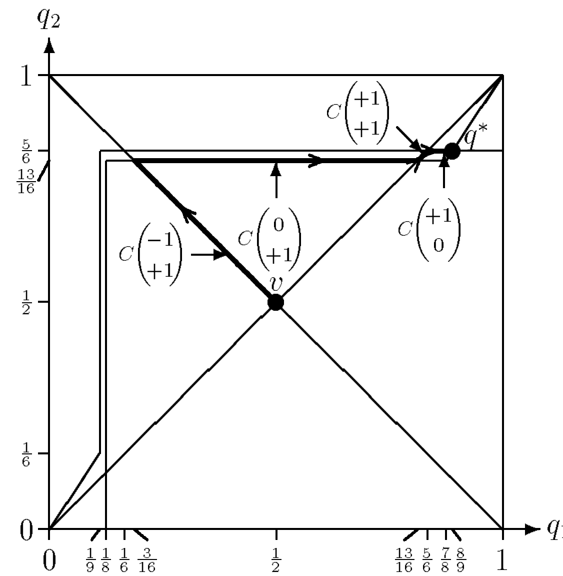

Figure VI

tighter. Notice that this initial adjustment of q_1 and q_2 does not go in the direction of the unique Drèze equilibrium. When q_2 equals $\frac{7}{10}$ rationing on the demand of commodity 2 becomes binding for consumer 1. This decreases the demand for commodity 2 and increases the demand for commodity 1 of consumer 1, since this consumer needs to supply less of commodity 1 in order to buy the maximal amount possible of commodity 2. Hence an increase in q_2 brings both market 1 and market 2 closer to equilibrium. When q_2 reaches the value $\frac{13}{16}$ market 1 attains an equilibrium situation. Now market 1 is kept in equilibrium and q_1 is no longer required to remain minimal relative to the initial state. The adjustment process follows a path in the set $C((0,+1)^\top)$. When q_1 reaches the value $\frac{13}{16}$, both q_1 and q_2 will be increased maximally relative to the initial state and market 1 is no longer kept in equilibrium. The adjustment process follows a path in the set $C((+1,+1)^\top)$. The increase in q_2 results in more demand rationing on the market of commodity 2, less demand of consumer 1 for commodity 2, and more demand of consumer 1 for commodity 1. The demands of the other consumers are not affected by the changes in q_1 and q_2. At $q = (\frac{5}{6}, \frac{5}{6})^\top$ the market of commodity 2 is in equilibrium. Market 2 is kept in equilibrium by keeping q_2 equal to $\frac{5}{6}$ and the adjustment process follows a path in the set $C((+1,0)^\top)$. Finally, the Drèze equilibrium $q^* = (\frac{8}{9}, \frac{5}{6})$ is reached.

In the next section it is proved that generically the quantity adjustment process is convergent. Even if the starting point is close to some Drèze equilibrium, the adjustment process may converge to another equilibrium. So in the terminology of Saari and Simon (1978) or Saari (1985), Definition 3.2 corresponds to an effective or globally convergent mechanism, but not to a locally effective or locally convergent mechanism. The adjustment process can be followed numerically arbitrarily close using the product-ray algorithm described in Doup and Talman (1987) or the exponent-ray algorithm described in Doup, van den Elzen, and Talman (1987). These algorithms generate piecewise linear paths of points corresponding with the adjustment process for a piecewise linear approximation of the total excess demand function.

4 Global and Universal Stability

In this section consumption sets and utility functions $\{X^i, u^i\}_{i=1}^m$, a rationing system $(\tilde{l}, \tilde{L})$, a fixed price system p, and an initial state $v \in Q^n$ are given. Then an economy $\mathcal{E}_\omega = (\{X^i, u^i, \omega^i\}, (\tilde{l}, \tilde{L}), p)$ is obtained for every specification of initial endowments ω in the set Ω defined by $\Omega = \prod_{i=1}^m \mathbb{R}_{++}^{n+1}$. Hence the set of economies is parametrized by the set of initial endowments Ω. Let initial endowments $\omega \in \Omega$ be given. To make clear the dependence on the initial endowments, for every $s \in \mathcal{S}$, for every $q \in Q^n$, the notation of $B(s)$, $C(s)$, C, $(\tilde{l}(q), \tilde{L}(q))$, $\hat{d}^i(q)$, $\forall i \in I_m$, and $\hat{z}(q)$ is changed into $B_\omega(s)$, $C_\omega(s)$, C_ω, $(\tilde{l}(q,\omega), \tilde{L}(q,\omega))$, $\hat{d}^i(q,\omega)$, $\forall i \in I_m$, and $\hat{z}(q,\omega)$, respectively. In this section it will be shown that the quantity adjustment process is generically globally and universally convergent, i.e., it converges to a Drèze equilibrium for a large class of economies (universal convergence) given any initial state of the economy (global convergence). It will be shown that the process converges, except possibly for economies parametrized by a set of initial endowments with a closure in Ω of Lebesgue measure zero.

Assume that the functions $\tilde{l} : \mathbb{R}_+^n \times \Omega \to -\mathbb{R}_+^{mn}$ and $\tilde{L} : \mathbb{R}_+^n \times \Omega \to \mathbb{R}_+^{mn}$ can be extended to continuous functions $\tilde{\tilde{l}} : \mathbb{R}_+^n \times \prod_{i=1}^m \mathbb{R}_+^{n+1} \to \mathbb{R}^{mn}$ and $\tilde{\tilde{L}} : \mathbb{R}_+^n \times \prod_{i=1}^m \mathbb{R}_+^{n+1} \to \mathbb{R}^{mn}$, respectively, and assume that, for every $\omega \in \prod_{i=1}^m \mathbb{R}_+^{n+1}$, the functions $\tilde{\tilde{l}}(\cdot,\omega) : \mathbb{R}_+^n \to \mathbb{R}^{mn}$ and $\tilde{\tilde{L}}(\cdot,\omega) : \mathbb{R}_+^n \to \mathbb{R}^{mn}$ satisfy Assumption A5. As in Section 2 a function $(\hat{l}, \hat{L}) : Q^n \times \Omega \to -\mathbb{R}_+^{mn} \times \mathbb{R}_+^{mn}$ will be constructed in such a way that on the adjustment path all conditions of a Drèze equilibrium are satisfied, except possibly the equality of demand and supply. Let a countable partitioning $\{\Omega(\nu) \mid \nu \in \mathbb{N}\}$ of Ω be given, which is locally finite, i.e., every $\omega \in \Omega$ has a neighborhood in Ω which intersects only finitely many sets $\Omega(\nu)$. Moreover, let the set $\Omega(\nu)$ be bounded for every $\nu \in \mathbb{N}$. The function $(\hat{l}, \hat{L})$ is constructed in such a way that, for every $\nu \in \mathbb{N}$, for every $(q,\omega), (q,\omega') \in Q^n \times \Omega(\nu)$, it holds that $(\hat{l}(q,\omega), \hat{L}(q,\omega)) = (\hat{l}(q,\omega'), \hat{L}(q,\omega'))$. A function $(\hat{l}, \hat{L})$ with this property will be called locally constant. A locally constant function $(\hat{l}, \hat{L})$ can be obtained as follows. Let some $\nu \in \mathbb{N}$ be given. For every $j \in I_n$, $\underline{q}_j$ is chosen such that $\forall \omega \in \Omega(\nu)$, $\forall q_j^1 \geq \underline{q}_j$, $\tilde{l}_j^i(q^1,\omega) < -\omega_j^i$, $\forall i \in I_m$, and $\overline{q}_j$ is chosen such that $\forall q_j^2 \geq \overline{q}_j$, $\tilde{L}_j^i(q^2,\omega) > \frac{p \cdot \omega^i}{p_j} - \omega_j^i$, $\forall i \in I_m$. Notice that the assumptions on $\Omega(\nu)$ and $(\tilde{l}, \tilde{L})$ guarantee that, for every $j \in I_n$, $\underline{q}_j$ and $\overline{q}_j$ with the above properties exist. Now, define

$$(\hat{l}(q,\omega), \hat{L}(q,\omega)) = (\tilde{l}(2\underline{q}_1 q_1, \ldots, 2\underline{q}_n q_n, \omega), \tilde{L}(2\overline{q}_1(1-q_1), \ldots, 2\overline{q}_n(1-q_n), \omega)),\ \forall q \in Q^n,\ \forall \omega \in \Omega(\nu).$$

Then, for every consumer $i \in I_m$, $\widehat{l}^i_j(q,\omega)$ is not binding for any $\omega \in \Omega(\nu)$ if $q_j \geq \frac{1}{2}$, and $\widehat{L}^i_j(q,\omega)$ is not binding for any $\omega \in \Omega(\nu)$ if $q_j \leq \frac{1}{2}$. A function $(\widehat{l},\widehat{L})$ with these properties will be called frictionless. For every $\nu \in \mathbb{N}$, let $\overline{\omega}^\nu$ be an element of $\Omega(\nu)$. It is useful to define, for every $\nu \in \mathbb{N}$, the function $(\widehat{l}^\nu,\widehat{L}^\nu) : Q^n \to -\mathbb{R}^{mn}_+ \times \mathbb{R}^{mn}_+$ by

$$(\widehat{l}^\nu(q),\widehat{L}^\nu(q)) = (\widehat{l}(q,\overline{\omega}^\nu),\widehat{L}(q,\overline{\omega}^\nu)),\ \forall q \in Q^n.$$

From now on the function $(\widehat{l},\widehat{L}) : Q^n \times \Omega \to -\mathbb{R}^{mn}_+ \times \mathbb{R}^{mn}_+$ is assumed to be given. In this section the following assumptions will be made.

B1. For every $i \in I_m$, the consumption set X^i is equal to $\mathbb{R}^{n+1}_{++}$.

B2. For every $i \in I_m$, the utility function $u^i : X^i \to \mathbb{R}$ is strictly increasing, strictly quasi-concave, three times continuously differentiable, the indifference surfaces of u^i have non-zero Gaussian curvature at every $x^i \in X^i$, and the closure of the indifference surfaces in $\mathbb{R}^{n+1}$ is a subset of $\mathbb{R}^{n+1}_{++}$.

B3. The price system p is an element of $\mathbb{R}^n_{++} \times \{1\}$.

B4. The function $(\widehat{l},\widehat{L}) : Q^n \times \Omega \to -\mathbb{R}^{mn}_+ \times \mathbb{R}^{mn}_+$ is locally constant, frictionless, twice continuously differentiable on Q^n and satisfies, for every $\nu \in \mathbb{N}$, for every $i \in I_m$, for every $j \in I_n$, for every $\tilde{q},\tilde{r} \in \mathbb{R}^n_+$

$$\begin{array}{ll}
\widehat{l}^{\nu,i}_j(\tilde{q}) = \widehat{l}^{\nu,i}_j(\tilde{r}) \text{ if } \tilde{q}_j = \tilde{r}_j, & \widehat{L}^{\nu,i}_j(\tilde{q}) = \widehat{L}^{\nu,i}_j(\tilde{r}) \text{ if } \tilde{q}_j = \tilde{r}_j, \\
\widehat{l}^{\nu,i}_j(\tilde{q}) = 0 \text{ if } \tilde{q}_j = 0, & \widehat{L}^{\nu,i}_j(\tilde{q}) = 0 \text{ if } \tilde{q}_j = 1, \\
\sum_{i\in I_m} \partial_{q_j}\widehat{l}^{\nu,i}_j(\tilde{q}) < 0, & \sum_{i\in I_m} \partial_{q_j}\widehat{L}^{\nu,i}_j(\tilde{q}) < 0.
\end{array}$$

Although Assumption B2 is not implied by Assumption A2, it can easily be shown that Theorems 2.2 and 3.1 remain valid under the Assumptions B1-B4 for any economy $\mathcal{E}_\omega$ with initial endowments $\omega \in \Omega$. Assumption B4 admits many possible rationing systems. In fact, any rationing system $(\tilde{l},\tilde{L})$ such that $\tilde{l}$ and $\tilde{L}$ can be extended to continuous functions $\tilde{\tilde{l}} : \mathbb{R}^n_+ \times \prod_{i=1}^m \mathbb{R}^{n+1}_+ \to \mathbb{R}^{mn}$ and $\tilde{\tilde{L}} : \mathbb{R}^n_+ \times \prod_{i=1}^m \mathbb{R}^{n+1}_+ \to \mathbb{R}^{mn}$, respectively, and such that, for every $\omega \in \prod_{i=1}^m \mathbb{R}^{n+1}_+$, the functions $\tilde{\tilde{l}}(\cdot,\omega) : \mathbb{R}^n_+ \to \mathbb{R}^{mn}$ and $\tilde{\tilde{L}}(\cdot,\omega) : \mathbb{R}^n_+ \to \mathbb{R}^{mn}$ satisfy Assumption A5, while, for every $\omega \in \Omega$, the functions $\tilde{l}(\cdot,\omega) : \mathbb{R}^n_+ \to -\mathbb{R}^{mn}_+$ and $\tilde{L}(\cdot,\omega) : \mathbb{R}^n_+ \to \mathbb{R}^{mn}_+$ are twice continuously differentiable and satisfy, for every $\tilde{q} \in \mathbb{R}^n_+$, for every $j \in I_n$, $\sum_{i\in I_m} \partial_{q_j}\tilde{l}^i_j(\tilde{q},\omega) < 0$ and $\sum_{i\in I_m} \partial_{q_j}\tilde{L}^i_j(\tilde{q},\omega) > 0$, yields a function $(\widehat{l},\widehat{L})$ satisfying Assumption B4 by the construction given in this section.

For every $\nu \in \mathbb{N}$, let $\overline{\omega}^\nu = \nu 1^{m(n+1)}$, define $\Omega(1) = \{\omega \in \Omega \mid \omega \leq \overline{\omega}^1\}$, and, for every $\nu \in \mathbb{N} \setminus \{1\}$, define $\Omega(\nu) = \{\omega \in \Omega \mid \omega \leq \overline{\omega}^\nu\} \setminus \Omega(\nu-1)$. Notice that $\{\Omega(\nu) \mid \nu \in \mathbb{N}\}$ is a locally finite partitioning of Ω such that $\Omega(\nu)$ is bounded for every $\nu \in \mathbb{N}$. First, a function $(\hat{l}, \hat{L})$ satisfying Assumption B4 corresponding to the uniform rationing system is given. Let some $\varepsilon > 0$ be given. For every $\nu \in \mathbb{N}$, for every $i \in I_m$, for every $j \in I_n$, define

$$\begin{aligned}
\hat{l}_j^{\nu,i}(q) &= -2(\nu + \varepsilon)q_j, \ \forall q \in Q^n, \\
\hat{L}_j^{\nu,i}(q) &= \frac{2\nu p \cdot 1^n}{p_j}(1 - q_j), \ \forall q \in Q^n.
\end{aligned}$$

Secondly, a function $(\hat{l}, \hat{L})$ satisfying Assumption B4 corresponding to the system where rationing is determined by priority is given. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(t) = e^{-\frac{1}{t}}$, $\forall t > 0$, and $\varphi(t) = 0$, $\forall t \leq 0$. Then φ is a C^∞ function. Let $\sigma_j : I_m \to I_m$ be a permutation specifying the order in which consumers are rationed on their excess supply on market $j \in I_n$. Let $\tau_j : I_m \to I_m$ be a permutation specifying the order in which consumers are rationed on their excess demand on market $j \in I_n$. Let some $\varepsilon > 0$ be given. For every $\nu \in \mathbb{N}$, for every $i \in I_m$, for every $j \in I_n$, define

$$\begin{aligned}
\hat{l}_j^{\nu,i}(q) &= -2m(\nu + \varepsilon)q_j, \ \forall q \in Q^n, \ \text{if } \sigma_j(i) = m, \\
\hat{l}_j^{\nu,i}(q) &= -2e\nu\varphi(\sigma_j(i) - m + 2mq_j), \ \forall q \in Q^n, \ \text{if } \sigma_j(i) \leq m - 1, \\
\hat{L}_j^{\nu,i}(q) &= \frac{2m\nu p \cdot 1^n}{p_j}(1 - q_j), \ \forall q \in Q^n, \ \text{if } \tau_j(i) = m, \\
\hat{L}_j^{\nu,i}(q) &= \frac{2e\nu p \cdot 1^n}{p_j}\varphi(\sigma_j(i) + m - 2mq_j), \ \forall q \in Q^n, \ \text{if } \tau_j(i) \leq m - 1.
\end{aligned}$$

A sign vector $r \in \mathbb{R}^{mn}$ determines the rationing state of every consumer $i \in I_m$ on every market $j \in I_n$, i.e., $r_j^i = -1$ indicates that consumer i is rationed on his supply on market j, $r_j^i = 0$ implies that consumer i is not rationed on market j, and $r_j^i = +1$ indicates that consumer i is rationed on his demand on market j. Since by Condition 3 of Definition 2.1 it can not occur that on any market there are both consumers rationed on their excess supply and consumers rationed on their excess demand, it is sufficient to consider sign vectors r such that given some market $j \in I_n$ either, for every $i \in I_m$, $r_j^i \geq 0$, or, for every $i \in I_m$, $r_j^i \leq 0$. Hence, define the set of feasible sign vectors $\mathcal{R}$ by

$$\mathcal{R} = \{r \in \mathbb{R}^{mn} \mid \forall (i,j) \in I_m \times I_n, \ r_j^i \in \{-1, 0, +1\}, \ \text{and } \forall j \in I_n, \ r_j^i \geq 0, \ \forall i \in I_m, \ \text{or } r_j^i \leq 0, \ \forall i \in I_m\}.$$

For a sign vector $r \in \mathcal{R}$, define the sets $I^-(r) = \{(i,j) \in I_m \times I_n \mid r_j^i = -1\}$, $I^0(r) = \{(i,j) \in I_m \times I_n \mid r_j^i = 0\}$, and $I^+(r) = \{(i,j) \in I_m \times I_n \mid r_j^i = +1\}$. Let $i^-(r)$, $i^0(r)$, and $i^+(r)$ denote the number of elements in the sets $I^-(r)$, $I^0(r)$, and $I^+(r)$, respectively. A sign vector $(r,s) \in \mathbb{R}^{mn} \times \mathbb{R}^n$ will be used to describe both the rationing state on the markets and the state of the markets concerning the total excess demand. It will be shown in Theorem 4.1 that it is sufficient to restrict attention to the set of feasible sign vectors $\mathcal{T} \subset \mathbb{R}^{mn} \times \mathbb{R}^n$ defined by

$$\mathcal{T} = \{(r,s) \in \mathcal{R} \times \mathcal{S} \mid \forall j \in I_n, \ s_j = -1 \Rightarrow \exists h \in I_m, \ r_j^h \neq +1,$$

$$\begin{aligned}
&\forall j \in I_n,\ s_j = 0 \text{ and } v_j < 1 \Rightarrow \exists h \in I_m,\ r_j^h \neq +1,\\
&\forall j \in I_n,\ s_j = 0 \text{ and } v_j > 0 \Rightarrow \exists h \in I_m,\ r_j^h \neq -1,\\
&\forall j \in I_n,\ s_j = +1 \Rightarrow \exists h \in I_m,\ r_j^h \neq -1\}.
\end{aligned}$$

For a sign vector $(r,s) \in \mathcal{T}$ and initial endowments $\omega \in \Omega$ define the set $C_\omega(r,s)$ by

$$\begin{aligned}
C_\omega(r,s) \;=\; \{q \in C_\omega(s) \mid\, & \forall (i,j) \in I^-(r),\ \widehat{d}^i_j(q,\omega) - \omega^i_j = \widehat{l}^i_j(q,\omega),\\
& \forall (i,j) \in I^0(r),\ \text{neither } \widehat{l}^i_j(q,\omega) \text{ nor } \widehat{L}^i_j(q,\omega) \text{ is binding},\\
& \forall (i,j) \in I^+(r),\ \widehat{d}^i_j(q,\omega) - \omega^i_j = \widehat{L}^i_j(q,\omega)\}.
\end{aligned}$$

Notice that in the case $\widehat{l}^i_j(q,\omega)$ ($\widehat{L}^i_j(q,\omega)$) is non-binding, it is still possible that $\widehat{d}^i_j(q,\omega) - \omega^i_j = \widehat{l}^i_j(q,\omega)$ ($\widehat{L}^i_j(q,\omega)$). First it is shown that there is indeed no loss of generality in considering only the feasible sign vectors $(r,s) \in \mathcal{T}$.

Theorem 4.1

For $\omega \in \Omega$, let the economy $\mathcal{E}_\omega = (\{X^i, u^i, \omega^i\}_{i=1}^m, (\widehat{l}, \widehat{L}), p)$ satisfying Assumptions B1-B4 and the initial state $v \in Q^n$ be given. Then $C_\omega = \cup_{(r,s)\in\mathcal{T}} C_\omega(r,s) \cup \{v \mid \widehat{z}(v,\omega) = 0^{n+1}$ and $\forall j \in I_n,\ v_j = 0$ or $v_j = 1\}$.

Proof

Clearly, $\cup_{(r,s)\in\mathcal{T}} C_\omega(r,s) \cup \{v \mid \widehat{z}(v,\omega) = 0^{n+1}$ and $\forall j \in I_n,\ v_j = 0$ or $v_j = 1\} \subset C_\omega$. To prove the converse, let some $\widetilde{q} \in C_\omega$ be given. Hence $\widetilde{q} \in C_\omega(\widetilde{s})$, for some $\widetilde{s} \in \mathcal{S}$. Exactly one of the following three statements is true. For every $(i,j) \in I_m \times I_n$, $\widehat{l}^i_j(\widetilde{q},\omega) = \widehat{d}^i_j(\widetilde{q},\omega) - \omega^i_j$, or $\widehat{l}^i_j(\widetilde{q},\omega) < \widehat{d}^i_j(\widetilde{q},\omega) - \omega^i_j < \widehat{L}^i_j(\widetilde{q},\omega)$, or $\widehat{d}^i_j(\widetilde{q},\omega) - \omega^i_j = \widehat{L}^i_j(\widetilde{q},\omega)$. Define $\widetilde{r} \in \mathcal{R}$ by $\widetilde{r}^i_j = -1$, $\widetilde{r}^i_j = 0$, or $\widetilde{r}^i_j = +1$, if the first, second, or third statement is true, respectively. In case $(\widetilde{r},\widetilde{s}) \in \mathcal{T}$, then $\widetilde{q} \in C_\omega(\widetilde{r},\widetilde{s})$. In case $(\widetilde{r},\widetilde{s}) \notin \mathcal{T}$, then at least one of the following four cases occurs.

1. $J^1 = \{j \in I_n \mid \widetilde{s}_j = -1,\ \forall i \in I_m,\ \widetilde{r}^i_j = +1\} \neq \emptyset$. Consider some $j \in J^1$. Since $\widetilde{r}^i_j = +1$ for every $i \in I_m$ it holds that $\widehat{z}_j(\widetilde{q},\omega) = \sum_{i=1}^m \widehat{L}^i_j(\widetilde{q},\omega) \geq 0$, while $\widetilde{s}_j = -1$ implies $\widehat{z}_j(\widetilde{q},\omega) \leq 0$. Consequently $\widehat{z}_j(\widetilde{q},\omega) = 0$ and by Assumption B4, $\widetilde{q}_j = 1$. Since $\widetilde{s}_j = -1$, $\exists \mu \in [0,1]$ such that $\widetilde{q}_j = \mu v_j$. Hence $\mu = 1$ and $v_j = 1$.

2. $J^2 = \{j \in I_n \mid v_j < 1,\ \widetilde{s}_j = 0,\ \forall i \in I_m,\ \widetilde{r}^i_j = +1\} \neq \emptyset$. Consider some $j \in J^2$. As in Case 1 this implies $\widetilde{q}_j = 1$. Since $\widetilde{s}_j = 0$ it holds that $0 = 1 - \widetilde{q}_j \geq \mu(1 - v_j)$, for some $\mu \in [0,1]$. Since $v_j < 1$ this implies $\mu = 0$ and $1 - \widetilde{q}_j = \mu(1 - v_j)$.

3. $J^3 = \{j \in I_n \mid v_j > 0,\ \widetilde{s}_j = 0,\ \forall i \in I_m,\ \widetilde{r}^i_j = -1\} \neq \emptyset$. Consider some $j \in J^3$. Since $\widetilde{r}^i_j = -1$ for every $i \in I_m$, it holds that $\widehat{z}_j(\widetilde{q},\omega) = \sum_{i=1}^m \widehat{l}^i_j(\widetilde{q},\omega)$. Since $\widetilde{s}_j = 0$ implies $\widehat{z}_j(\widetilde{q},\omega) = 0$, it follows from Assumption B4 that $\widetilde{q}_j = 0$. Since $\widetilde{q}_j \geq \mu v_j$ and $v_j > 0$ it follows also that $\mu = 0$ and $\widetilde{q}_j = \mu v_j$.

4. $J^4 = \{j \in I_n \mid \widetilde{s}_j = +1,\ \forall i \in I_m,\ \widetilde{r}^i_j = -1\} \neq \emptyset$. Consider some $j \in J^4$. As in Case 3 it follows that $\widetilde{q}_j = 0$. Since $\widetilde{s}_j = +1$, $1 = 1 - \widetilde{q}_j = \mu(1 - v_j)$, for some $\mu \in [0,1]$. Consequently,

$\mu = 1$ and $v_j = 0$.
Define $\widehat{s} \in \mathbb{R}^n$ by $\widehat{s}_j = 0$, $\forall j \in J^1$, $\widehat{s}_j = +1$, $\forall j \in J^2$, $\widehat{s}_j = -1$, $\forall j \in J^3$, $\widehat{s}_j = 0$, $\forall j \in J^4$, and $\widehat{s}_j = \widetilde{s}_j$, $\forall j \in I_n \setminus (J_1 \cup J_2 \cup J_3 \cup J_4)$. Then $\widehat{s} \in \mathcal{S}$, unless $\widehat{s} = 0^n$. In the case $\widehat{s} \neq 0^n$ it is easily verified that $(\widetilde{r}, \widehat{s}) \in \mathcal{T}$ and $\widetilde{q} \in C_\omega(\widetilde{r}, \widehat{s})$.
Next consider the case $\widehat{s} = 0^n$. This implies $J^1 \cup J^4 \neq \emptyset$ and therefore $\mu = 1$, $\widetilde{q} = v$, and $\widehat{z}(v, \omega) = 0^{n+1}$. When there exists some $j' \in I_n \setminus (J^1 \cup J^4)$ with $0 < v_{j'} < 1$, then, since $j' \notin J^1 \cup J^4$, $\exists i' \in I_m$ such that $\widetilde{r}^{i'}_{j'} \neq -1$, and $\exists i'' \in I_m$ such that $\widetilde{r}^{i''}_{j'} \neq +1$. Define $\overline{s} \in \mathcal{S}$ by $\overline{s}_{j'} = -1$, and $\overline{s}_j = \widehat{s}_j$, $\forall j \in I_n \setminus \{j'\}$. Clearly, $(\widetilde{r}, \overline{s}) \in \mathcal{T}$ and $\widetilde{q} \in C_\omega(\widetilde{r}, \overline{s})$. When there exists no $j' \in I_n \setminus (J^1 \cup J^4)$ with $0 < v_{j'} < 1$, then it holds that $\widetilde{q} = v$, $\widehat{z}(v, \omega) = 0^{n+1}$, and, for every $j \in I_n$, $v_j = 0$ or $v_j = 1$. Q.E.D.

It will be shown that, for almost every $\omega \in \Omega$, for every $(r, s) \in \mathcal{T}$, $C_\omega(r, s)$ is a 1-dimensional C^2 manifold with boundary. Some definitions are given first.

For some $t \geq 1$ a subset M of $\mathbb{R}^N$ is called a k-dimensional C^t manifold with generalized boundary (MGB), if for every $\widetilde{x} \in M$ there exists a local C^t coordinate system of $\mathbb{R}^N$ around $\widetilde{x}$, i.e., a C^t diffeomorphism $\varphi : U \to V$ where U is an open subset of $\mathbb{R}^N$ containing $\widetilde{x}$ and V is open in $\mathbb{R}^N$, and some $b(\widetilde{x}) \in \{0\} \cup I_k$ such that $\varphi(\widetilde{x}) = 0^N$ and $\varphi(U \cap M)$ equals

$$\{y \in V \mid y_1 = \cdots = y_{N-k} = 0,\ y_{N-k+1} \geq 0, \ldots, y_{N-k+b(\tilde{x})} \geq 0\}.$$

If for every element $\widetilde{x}$ of an MGB M, $b(\widetilde{x}) \leq 1$, then M is called a manifold with boundary and it is easily shown that the set of elements $\widetilde{x}$ for which $b(\widetilde{x}) = 1$ is a $(k-1)$-dimensional manifold, called the boundary of M. Let X be some open subset of $\mathbb{R}^N$, let J^1 and J^2 be two finite, possibly empty, index sets and let g_j, $\forall j \in J^1$, and h_j, $\forall j \in J^2$, be C^t functions defined on X. Define

$$M[g, h] = \{x \in X \mid \forall j \in J^1,\ g_j(x) = 0,\ \forall j \in J^2,\ h_j(x) \geq 0\}.$$

For every $x \in X$, define $J^0(x) = \{j \in J^2 \mid h_j(x) = 0\}$. If for every $\widetilde{x} \in M[g, h]$ it holds that $\{\partial_x g_j(\widetilde{x})^\top \in \mathbb{R}^N \mid \forall j \in J^1\} \cup \{\partial_x h_j(\widetilde{x})^\top \in \mathbb{R}^N \mid \forall j \in J^0(\widetilde{x})\}$ is a set of independent vectors then $M[g, h]$ is called a C^t regular constraint set (RCS). In Jongen, Jonker, and Twilt (1983, Lemma 3.1.2, Example 3.1.3) it is shown that every C^t RCS is a $(N - |J^1|)$-dimensional C^t MGB with for every $x \in M[g, h]$, $b(x) = |J^0(x)|$.

Let some $\nu \in \mathbb{N}$ and some $(r, s) \in \mathcal{T}$ be given. A system of equalities and inequalities describing $C_\omega(r, s)$ for every $\omega \in \Omega(\nu)$ will be given. Using this system it will be shown in Theorem 4.6 that $C_\omega(r, s)$ is an RCS for almost every $\omega \in \Omega(\nu)$. Define the set $J(r, s)$ by

$$J(r, s) = \{j \in I_n \mid s_j = -1,\ \forall i \in I_m,\ r^i_j = -1\} \cup \{j \in I_n \mid s_j = +1,\ \forall i \in I_m,\ r^i_j = +1\},$$

and let $j(r, s)$ be the lowest ranked element in the set $J(r, s)$ if $J(r, s)$ is non-empty and let $j(r, s) = 0$ if $J(r, s) = \emptyset$. The set $J(r, s)$ contains the markets where either the supply of all consumers is rationed, or the demand of all consumers is rationed. Define the sets

$$\begin{aligned} J^-(r,s) &= I^-(s) \cap (\{j \in I_n \mid \exists i' \in I_m,\ r_j^{i'} \neq -1\} \cup \{j(r,s)\}), \text{ and} \\ J^+(r,s) &= I^+(s) \cap (\{j \in I_n \mid \exists i' \in I_m,\ r_j^{i'} \neq +1\} \cup \{j(r,s)\}). \end{aligned}$$

Moreover, define the sets

$$\begin{aligned} K^-(r,s) &= I^0(s) \cap \{j \in I_n \mid v_j > 0 \text{ and } \exists i' \in I_m,\ r_j^{i'} \neq +1\}, \\ K^+(r,s) &= I^0(s) \cap \{j \in I_n \mid v_j < 1 \text{ and } \exists i' \in I_m,\ r_j^{i'} \neq -1\}. \end{aligned}$$

These sets are needed to formulate the system of equalities and inequalities mentioned above in such a way that no equality or inequality is redundant.

Let some $\nu \in \mathbb{N}$, some $\omega \in \Omega(\nu)$, and some $(r,s) \in \mathcal{T}$ be given. In Theorem 4.2 it is shown that $q \in C_\omega(r,s)$ if and only if $q \in \mathbb{R}^n$ and there exists an element $(x,\lambda,\mu) \in \mathbb{R}^{m(n+1)}_{++} \times \mathbb{R}^m \times \mathbb{R}$ such that (x,λ,q,μ) satisfies

$$\begin{aligned}
\partial_{x_j^i} u^i(x^i) - \lambda^i p_j &= 0,\ \forall (i,j) \in I^0(r), && (4)\\
\partial_{x_{n+1}^i} u^i(x^i) - \lambda^i &= 0,\ \forall i \in I_m, && (5)\\
p \cdot (x^i - \omega^i) &= 0,\ \forall i \in I_m, && (6)\\
x_j^i - \omega_j^i - \widehat{l}_j^{\nu,i}(q) &= 0,\ \forall (i,j) \in I^-(r), && (7)\\
-x_j^i + \omega_j^i + \widehat{L}_j^{\nu,i}(q) &= 0,\ \forall (i,j) \in I^+(r), && (8)\\
\sum_{i=1}^m (x_j^i - \omega_j^i) &= 0,\ \forall j \in I^0(s), && (9)\\
q_j - \mu v_j &= 0,\ \forall j \in I^-(s), && (10)\\
(1-q_j) - \mu(1-v_j) &= 0,\ \forall j \in I^+(s), && (11)\\
-\partial_{x_j^i} u^i(x^i) + \lambda^i p_j &\geq 0,\ \forall (i,j) \in I^-(r), && (12)\\
\partial_{x_j^i} u^i(x^i) - \lambda^i p_j &\geq 0,\ \forall (i,j) \in I^+(r), && (13)\\
x_j^i - \omega_j^i - \widehat{l}_j^{\nu,i}(q) &\geq 0,\ \forall (i,j) \in I^0(r), && (14)\\
-x_j^i + \omega_j^i + \widehat{L}_j^{\nu,i}(q) &\geq 0,\ \forall (i,j) \in I^0(r), && (15)\\
\sum_{i=1}^m (-x_j^i + \omega_j^i) &\geq 0,\ \forall j \in J^-(r,s), && (16)\\
\sum_{i=1}^m (x_j^i - \omega_j^i) &\geq 0,\ \forall j \in J^+(r,s), && (17)\\
q_j - \mu v_j &\geq 0,\ \forall j \in K^-(r,s), && (18)\\
(1-q_j) - \mu(1-v_j) &\geq 0,\ \forall j \in K^+(r,s), && (19)\\
1 - \mu &\geq 0. && (20)
\end{aligned}$$

Notice that, for every $\nu \in \mathbb{N}$, $\widehat{l}^\nu$ and $\widehat{L}^\nu$ are assumed to be defined also outside Q^n. Any twice continuously differentiable extension of $\widehat{l}^\nu$ and $\widehat{L}^\nu$ defined on $\mathbb{R}^n$ such that, for every

$\tilde{q} \in \mathbb{R}^n \setminus Q^n$, $\hat{l}^\nu(\tilde{q}) \leq \hat{L}^\nu(\tilde{q})$, $\sum_{i=1}^m \partial_{q_j} \hat{l}_j^{\nu,i}(\tilde{q}) < 0$, and $\sum_{i=1}^m \partial_{q_j} \hat{L}_j^{\nu,i}(\tilde{q}) < 0$, suffices. Clearly, such an extension exists if $(\hat{l}, \hat{L})$ satisfies Assumption B4. In the following, let N denote $m(n+1)+m+n$, the number of variables in (x, λ, q, μ) minus one, and let N' denote $2m(n+1)+m+n+1$, the total number of variables in $(x, \lambda, \omega, q, \mu)$. Define the function $\hat{\mu} : Q^n \to [0,1]$ by $\mu(q) = \min\{\min\{\frac{q_j}{v_j} \mid j \in I_n \text{ with } v_j > 0\}, \min\{\frac{1-q_j}{1-v_j} \mid j \in I_n \text{ with } v_j < 1\}\}$, $\forall q \in Q^n$. Define the function $f : Q^n \times \Omega \to \mathbb{R}^{N+1}$ by

$$f(q,\omega) = ((\hat{d}^i(q,\omega))_{i=1}^m, (\partial_{x_{n+1}^i} u^i(\hat{d}^i(q,\omega)))_{i=1}^m, q, \hat{\mu}(q)),\ \forall q \in Q^n,\ \forall \omega \in \Omega.$$

Theorem 4.2

Let $(\{X^i, u^i\}_{i=1}^m, (\hat{l}, \hat{L}), p)$ satisfying Assumptions B1-B4 and $v \in Q^n$ be given. Moreover, let some $\nu \in \mathbb{N}$, some $\omega \in \Omega(\nu)$, and some $(r,s) \in \mathcal{T}$ be given. Then $\tilde{q} \in C_\omega(r,s)$ and $(\tilde{x}, \tilde{\lambda}, \tilde{q}, \tilde{\mu}) = f(\tilde{q}, \omega)$ if and only if $(\tilde{x}, \tilde{\lambda}, \tilde{q}, \tilde{\mu}) \in \mathbb{R}_{++}^{m(n+1)} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ and $(\tilde{x}, \tilde{\lambda}, \tilde{q}, \tilde{\mu})$ satisfies (4)-(20).

Proof

If $\tilde{q} \in C_\omega(r,s)$ and $(\tilde{x}, \tilde{\lambda}, \tilde{q}, \tilde{\mu}) = f(\tilde{q}, \omega)$, then it is clear that $(\tilde{x}, \tilde{\lambda}, \tilde{q}, \tilde{\mu}) \in \mathbb{R}_{++}^{m(n+1)} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ satisfies (4)-(20). To show the converse, let some $(\tilde{x}, \tilde{\lambda}, \tilde{q}, \tilde{\mu}) \in \mathbb{R}_{++}^{m(n+1)} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ satisfying (4)-(20) be given. First it is shown that, for every $j \in I_n$, $0 \leq \tilde{q}_j \leq 1$ and $0 \leq \tilde{\mu} \leq 1$. By (20) it holds that $\tilde{\mu} \leq 1$. By (10) and (11) it follows that

$$\tilde{q}_j = \tilde{\mu} v_j \leq 1,\ \forall j \in I^-(s), \tag{21}$$

$$\tilde{q}_j = 1 - \tilde{\mu}(1 - v_j) \geq 1 - 1 + v_j \geq 0,\ \forall j \in I^+(s). \tag{22}$$

Using $\hat{l}_j^{\nu,i}(\tilde{q}) \leq \hat{L}_j^{\nu,i}(\tilde{q})$ and (7), (8), and (14) yields $\sum_{i=1}^m \hat{l}_j^{\nu,i}(\tilde{q}) \leq \sum_{i=1}^m (\tilde{x}_j^i - \omega_j^i)$. For every $j \in I^0(s) \cup J^-(r,s)$ it follows by (9) and (16) that $\sum_{i=1}^m (\tilde{x}_j^i - \omega_j^i) \leq 0$. Consequently,

$$\tilde{q}_j \geq 0,\ \forall j \in I^0(s) \cup J^-(r,s). \tag{23}$$

Similarly, it follows that

$$\tilde{q}_j \leq 1,\ \forall j \in I^0(s) \cup J^+(r,s), \tag{24}$$

using (7), (8), (9), (15), and (17). Now, three cases have to be considered: $j(r,s) = 0$, $j(r,s) \in I^-(s)$, and $j(r,s) \in I^+(s)$.

1. If $j(r,s) = 0$, then $J^-(r,s) = I^-(s)$ and $J^+(r,s) = I^+(s)$. So (21)-(24) yield $0 \leq \tilde{q}_j \leq 1$, $\forall j \in I_n$. When $I^+(s) \neq \emptyset$, then consider some $j \in I^+(s)$. Clearly, $v_j < 1$ and therefore (11) implies $\tilde{\mu} \geq 0$. When $I^+(s) = \emptyset$, then $I^-(s) \neq \emptyset$, and now (10) implies $\tilde{\mu} \geq 0$. This together with (20) shows that $0 \leq \tilde{\mu} \leq 1$.

2. If $j(r,s) \in I^-(s)$, then (23) implies $\tilde{q}_{j(r,s)} \geq 0$. Since $j(r,s) \in I^-(s)$ implies $v_{j(r,s)} > 0$, (10) yields $\tilde{\mu} \geq 0$. Now it follows from (10) that $\tilde{q}_j \geq 0$, $\forall j \in I^-(s)$, and from (11) that $\tilde{q}_j \leq 1$, $\forall j \in I^+(s)$.

3. This case is similar to Case 2.

Equations (4)-(8) and inequalities (12)-(15) guarantee that $\tilde{x}^i = \hat{d}^i(\tilde{q}, \omega)$, $\forall i \in I_m$. Hence, (9) implies $\hat{z}_j(\tilde{q}, \omega) = 0$, $\forall j \in I^0(s)$. Consider some $j' \in I^-(s)$. Either, $\exists i' \in I_m$, $r^{i'}_{j'} \neq -1$, and by (16), $\hat{z}_{j'}(\tilde{q}, \omega) \leq 0$, or $\forall i \in I_m$, $r^i_{j'} = -1$, and therefore by (7), $\hat{z}_{j'}(\tilde{q}, \omega) = \sum_{i=1}^m (\tilde{x}^i_{j'} - \omega^i_{j'}) = \sum_{i=1}^m \hat{l}^i_{j'}(\tilde{q}, \omega) \leq 0$. Similarly, it can be shown that $\hat{z}_j(\tilde{q}, \omega) \geq 0$, $\forall j \in I^+(s)$. Hence $\tilde{q} \in B_\omega(s)$.

For every $j \in I^0(s)$ with $v_j = 0$ it holds that $\tilde{\mu} v_j \leq \tilde{q}_j$, and for every $j \in I^0(s)$ with $v_j = 1$ it holds that $\tilde{q}_j \leq 1 - \tilde{\mu} + \tilde{\mu} v_j$. For every $j \in I^0(s)$ such that, for all $i \in I_m$, $(i,j) \in I^+(r)$, (8) and (9) imply $\sum_{i=1}^m \hat{L}^{\nu,i}_j(\tilde{q}) = 0$, and therefore $\tilde{q}_j = 1$. Hence $\tilde{\mu} v_j \leq \tilde{q}_j$. For every $j \in I^0(s)$ such that, for all $i \in I_m$, $(i,j) \in I^-(r)$, (7) and (9) imply $\sum_{i=1}^m \hat{l}^{\nu,i}_j(\tilde{q}) = 0$, and therefore $\tilde{q}_j = 0$. Hence $\tilde{q}_j \leq 1 - \tilde{\mu} + \tilde{\mu} v_j$. Therefore, (10), (11), (18), and (19), together with $\tilde{q} \in B_\omega(s)$, imply that $\tilde{q} \in C_\omega(s)$. Now (4), (7), and (8) imply that for every $(i,j) \in I^0(r)$ neither $\hat{l}^i_j(\tilde{q}, \omega)$ nor $\hat{L}^i_j(\tilde{q}, \omega)$ is binding, that for every $(i,j) \in I^+(r)$ it holds that $\hat{d}^i_j(\tilde{q}, \omega) - \omega^i_j = \hat{L}^i_j(\tilde{q}, \omega)$, and that for every $(i,j) \in I^-(r)$ it holds that $\hat{l}^i_j(\tilde{q}, \omega) = \hat{d}^i_j(\tilde{q}, \omega) - \omega^i_j$, respectively. Hence $\tilde{q} \in C_\omega(r,s)$. Using the above and (5) it follows that $(\tilde{x}, \tilde{\lambda}, \tilde{q}, \tilde{\mu}) = f(\tilde{q}, \omega)$. Q.E.D.

Let $\nu \in \mathbb{N}$ and $(r,s) \in \mathcal{T}$ be given, and let $\hat{\Omega}(\nu) \subset \Omega$ be an open set containing the closure of $\Omega(\nu)$ in Ω. Define the function $\psi^{\nu,r,s} : \mathbb{R}^{m(n+1)}_{++} \times \mathbb{R}^m \times \hat{\Omega}(\nu) \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^N$ such that $\psi^{\nu,r,s}(x, \lambda, \omega, q, \mu)$ is the left-hand side of (4)-(11). For $\omega \in \hat{\Omega}(\nu)$, define $\psi^{\nu,r,s,\omega} : \mathbb{R}^{m(n+1)}_{++} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^N$ by $\psi^{\nu,r,s,\omega}(x, \lambda, q, \mu) = \psi^{\nu,r,s}(x, \lambda, \omega, q, \mu)$, $\forall (x, \lambda, q, \mu) \in \mathbb{R}^{m(n+1)}_{++} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$. The $N \times N'$ matrix of partial derivatives of $\psi^{\nu,r,s}$ evaluated at a point $\tilde{\xi} = (\tilde{x}, \tilde{\lambda}, \tilde{\omega}, \tilde{q}, \tilde{\mu})$ satisfying $\psi^{\nu,r,s}(\tilde{\xi}) = 0^N$ is denoted by M and is given in Table I. In this table 0^k is a row vector, and e^k_j is defined as the j-th k-dimensional unit vector, being defined as a row vector. In the following we will denote the indicator function of a set S by $\mathcal{I}_S$, so $\mathcal{I}_S(x) = 1$ if $x \in S$ and $\mathcal{I}_S(x) = 0$ if $x \notin S$.

Lemma 4.3

Let $(\{X^i, u^i\}_{i=1}^m, (\hat{l}, \hat{L}), p)$ satisfying Assumptions B1-B4 and $v \in Q^n$ be given. Moreover, let $\nu \in \mathbb{N}$ and $(r,s) \in \mathcal{T}$ be given. Then $\psi^{\nu,r,s,\omega} \pitchfork \{0^N\}$ and $\psi^{\nu,r,s,\omega^{-1}}(\{0^N\})$ is a 1-dimensional C^2 manifold, except for a set of initial endowments $\omega \in \hat{\Omega}(\nu)$ with Lebesgue measure zero.

Proof

First it is shown that $\psi^{\nu,r,s} \pitchfork \{0^N\}$. To do this, it has to be shown that the rows of M are independent. This is done by proving that $y^\top M = 0^{N'^\top}$ implies $y = 0^N$. Hence, let $y \in \mathbb{R}^N$ satisfy $y^\top M = 0^{N'^\top}$. Since in Table I M is subdivided into eight parts, the components of y will be denoted accordingly in the obvious way by $y_{(1,i,j)}$, $\forall (i,j) \in I^0(r)$, $y_{(2,i)}$, $\forall i \in I_m$, and so on. For every $i \in I_m$ it holds that $0 = y^\top \partial_{\omega^i_{n+1}} \psi^{\nu,r,s}(\tilde{\xi}) = -y_{(3,i)}$. Consequently,

$$y_{(3,i)} = 0, \ \forall i \in I_m. \tag{25}$$

	$\partial_x \psi^{\nu,r,s}$	$\partial_\lambda \psi^{\nu,r,s}$	$\partial_\omega \psi^{\nu,r,s}$	$\partial_q \psi^{\nu,r,s}$	$\partial_\mu \psi^{\nu,r,s}$	
1	$0^{(i-1)(n+1)},\ \partial^2_{x^i x^i_j} u^i(\tilde{x}^i),\ 0^{(m-i)(n+1)}$	$-p_j e^m_i$	$0^{m(n+1)}$	0^n	0	$(i,j) \in I^0(r$
2	$0^{(i-1)(n+1)},\ \partial^2_{x^i x^i_{n+1}} u^i(\tilde{x}^i),\ 0^{(m-i)(n+1)}$	$-e^m_i$	$0^{m(n+1)}$	0^n	0	$i \in I_m$
3	$0^{(i-1)(n+1)},\ p^\top,\ 0^{(m-i)(n+1)}$	0^m	$0^{(i-1)(n+1)},\ -p^\top,\ 0^{(m-i)(n+1)}$	0^n	0	$i \in I_m$
4	$e^{m(n+1)}_{(i-1)(n+1)+j}$	0^m	$-e^{m(n+1)}_{(i-1)(n+1)+j}$	$-\partial_{q_j} \widehat{l}^{\nu,i}_j(\tilde{q}) e^n_j$	0	$(i,j) \in I^-(r$
5	$-e^{m(n+1)}_{(i-1)(n+1)+j}$	0^m	$e^{m(n+1)}_{(i-1)(n+1)+j}$	$\partial_{q_j} \widehat{L}^{\nu,i}_j(\tilde{q}) e^n_j$	0	$(i,j) \in I^+(r$
6	$e^{n+1}_j, \ldots, e^{n+1}_j$	0^m	$-e^{n+1}_j, \ldots, -e^{n+1}_j$	0^n	0	$j \in I^0(s)$
7	$0^{m(n+1)}$	0^m	$0^{m(n+1)}$	e^n_j	$-v_j$	$j \in I^-(s)$
8	$0^{m(n+1)}$	0^m	$0^{m(n+1)}$	$-e^n_j$	$-(1-v_j)$	$j \in I^+(s)$
	$m(n+1)$	m	$m(n+1)$	n	1	

Table I. The matrix M.

Using (25) it holds for every $(i,j) \in I_m \times I_n$ that

$$0 = y^\top \partial_{\omega^i_j} \psi^{\nu,r,s}(\widetilde{\xi}) = -y_{(4,i,j)} \mathcal{I}_{I^-(r)}((i,j)) + y_{(5,i,j)} \mathcal{I}_{I^+(r)}((i,j)) - y_{(6,j)} \mathcal{I}_{I^0(s)}(j). \tag{26}$$

Consider an element $(i',j') \in I^-(r)$. Then, clearly, $(i',j') \notin I^+(r)$. There are three possibilities. Either $j' \notin I^0(s)$, implying $y_{(4,i',j')} = 0$. Or $j' \in I^0(s)$ and for some $i'' \in I_m$, $(i'',j') \notin I^-(r)$, implying by (26) with $(i,j) = (i'',j')$ that $y_{(6,j')} = 0$ and consequently, using (26) with $(i,j) = (i',j')$, that $y_{(4,i',j')} = 0$. Finally, consider the case with $(i,j') \in I^-(r)$, $\forall i \in I_m$, and $j' \in I^0(s)$. From (26) it follows that $y_{(6,j')} = -y_{(4,i,j')}$, $\forall i \in I_m$. Since $j' \in I^0(s)$ and therefore $j' \notin I^-(s) \cup I^+(s)$ it holds that $0 = y^\top \partial_{q_{j'}} \psi^{\nu,r,s}(\widetilde{\xi}) = -\sum_{i \in I_m} y_{(4,i,j')} \partial_{q_{j'}} \widehat{l}^{\nu,i}_{j'}(\widetilde{q}) = y_{(6,j')} \sum_{i \in I_m} \partial_{q_{j'}} \widehat{l}^{\nu,i}_{j'}(\widetilde{q})$, and since $\sum_{i \in I_m} \partial_{q_{j'}} \widehat{l}^{\nu,i}_{j'}(\widetilde{q}) < 0$ this implies $y_{(6,j')} = 0$. Hence $y_{(4,i,j')} = 0$, $\forall i \in I_m$. It has been shown now that

$$y_{(4,i,j)} = 0,\ \forall (i,j) \in I^-(r). \tag{27}$$

That

$$y_{(5,i,j)} = 0,\ \forall (i,j) \in I^+(r), \tag{28}$$

can be shown in a similar way. From (26) it then follows that

$$y_{(6,j)} = 0,\ \forall j \in I^0(s). \tag{29}$$

By (27) and (28) it holds for every $j \in I_n$ that

$$0 = y^\top \partial_{q_j} \psi^{\nu,r,s}(\widetilde{\xi}) = y_{(7,j)} \mathcal{I}_{I^-(s)}(j) - y_{(8,j)} \mathcal{I}_{I^+(s)}(j). \tag{30}$$

Since $I^-(s) \cap I^+(s) = \emptyset$ it follows that

$$y_{(7,j)} = 0,\ \forall j \in I^-(s), \tag{31}$$

$$y_{(8,j)} = 0,\ \forall j \in I^+(s). \tag{32}$$

Let some consumer $i' \in I_m$ be given. From the non-zero Gaussian curvature of the utility functions it follows that

$$\begin{vmatrix} \partial^2_{x^{i'} x^{i'}} u^{i'}(\widetilde{x}^{i'}) & \partial_{x^{i'}} u^{i'}(\widetilde{x}^{i'})^\top \\ \partial_{x^{i'}} u^{i'}(\widetilde{x}^{i'}) & 0 \end{vmatrix} \neq 0,$$

see for instance Mas-Colell (1985), Proposition 2.5.1. Hence, the rows of $[\partial^2_{x^{i'} x^{i'}} u^{i'}(\widetilde{x}^{i'})\ \partial_{x^{i'}} u^{i'}(\widetilde{x}^{i'})^\top]$ corresponding with the indices $j \in I_n$ satisfying $(i', j) \in I^0(r)$ and the index $n+1$ are independent. From this and the fact that $\partial_{x^{i'}_j} u^{i'}(\widetilde{x}^{i'}) = \widetilde{\lambda}^{i'} p_j$, $\forall (i', j) \in I^0(r)$, and for $j = n+1$, it follows that the first $i^0(r) + m$ rows of M are independent. By (25), (27), (28), (29), (31), and (32) it follows that

$$0^{N'^\top} = y^\top M = \sum_{(i,j) \in I^0(r)} y_{(1,i,j)} M_{((1,i,j),\cdot)} + \sum_{i \in I_m} y_{(2,i)} M_{((2,i),\cdot)}.$$

Consequently, $y_{(1,i,j)} = 0$, $\forall (i,j) \in I^0(r)$, and $y_{(2,i)} = 0$, $\forall i \in I_m$. It has been shown now that $y = 0^N$, and therefore $\psi^{\nu,r,s} \pitchfork \{0^N\}$. By the transversality theorem (see for example Theorem I.2.2 of Mas-Colell (1985)) and since $\psi^{\nu,r,s}$ is a twice continuously differentiable function it follows that the complement of the set $\{\omega \in \widehat{\Omega}(\nu) \mid \psi^{\nu,r,s,\omega} \pitchfork \{0^N\}\}$ has Lebesgue measure zero. Since $\psi^{\nu,r,s,\omega}$ maps from a manifold with dimension $N+1$ into a manifold with dimension N, and since $\psi^{\nu,r,s,\omega}$ is a twice continuously differentiable function, $\psi^{\nu,r,s,\omega} \pitchfork \{0^N\}$ implies that ${\psi^{\nu,r,s,\omega}}^{-1}(\{0^N\})$ is a 1-dimensional C^2 manifold. Q.E.D.

Let some $\nu \in \mathbb{N}$ and some $(r,s) \in \mathcal{T}$ be given. Let K be the set of indices corresponding with one of the inequalities in (12)-(20). More precisely, K is defined by

$$\begin{aligned} K \;=\; & I^-(r) \cup I^+(r) \cup \{(i,j,-) \mid (i,j) \in I^0(r), \nexists i' \in I_m,\ (i',j) \in I^+(r)\} \\ & \cup \{(i,j,+) \mid (i,j) \in I^0(r),\ \nexists i' \in I_m,\ (i',j) \in I^-(r)\} \cup J^-(r,s) \cup J^+(r,s) \\ & \cup \{(j,-) \mid j \in K^-(r,s)\} \cup \{(j,+) \mid j \in K^+(r,s)\} \cup \{0\}. \end{aligned}$$

Consider some index $k \in K$. When $k = (i,j) \in I^-(r)$, then (i,j) corresponds to inequality (i,j) in (12), $k = (i,j) \in I^+(r)$ corresponds to inequality (i,j) in (13), $k = (i,j,-)$ corresponds to inequality (i,j) in (14), $k = (i,j,+)$ corresponds to inequality (i,j) in (15), $k \in J^-(r,s)$ corresponds to inequality k in (16), $k \in J^+(r,s)$ corresponds to inequality k in (17), $k = (j,-)$ corresponds to inequality j in (18), $k = (j,+)$ corresponds to inequality j in (19), and $k = 0$ corresponds to inequality (20). Notice that not every $(i,j) \in I^0(r)$ yields indices $(i,j,-)$ and $(i,j,+)$. It will be sufficient to consider the indices in K. For every $k \in K$ define a function $\psi^{\nu,r,s}_k : \mathbb{R}^{m(n+1)}_{++} \times \mathbb{R}^m \times \widehat{\Omega}(\nu) \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{N+1}$ such that $\psi^{\nu,r,s}_k$ is the left-hand side of (4)-(11) and the inequality corresponding to k. Define, for every $\omega \in \widehat{\Omega}(\nu)$, the function $\psi^{\nu,r,s,\omega}_k$ in the obvious way.

Lemma 4.4

Let $(\{X^i, u^i\}_{i=1}^m, (\widehat{l}, \widehat{L}), p)$ satisfying Assumptions B1-B4 and $v \in Q^n$ be given. Moreover, let $\nu \in \mathbb{N}$, $(r,s) \in \mathcal{T}$, and $k \in K$ be given. Then, $\psi_k^{\nu,r,s,\omega} \pitchfork \{0^{N+1}\}$ and $\psi_k^{\nu,r,s,\omega^{-1}}(\{0^{N+1}\})$ is a 0-dimensional C^2 manifold, except for a set of initial endowments $\omega \in \widehat{\Omega}(\nu)$ with Lebesgue measure zero.

Proof

First it is shown that $\psi_k^{\nu,r,s} \pitchfork \{0^{N+1}\}$. Let $\psi_k^{\nu,r,s}(\widetilde{\xi}) = 0^{N+1}$. It will be shown that the rows of the $(N+1) \times N'$ matrix of partial derivatives of $\psi_k^{\nu,r,s}$ at $\widetilde{\xi}$, denoted by M_k, are independent. Let $y \in \mathbb{R}^{N+1}$ satisfy $y^\top M_k = 0^{N'^\top}$. It will be shown that $y = 0^{N+1}$. Let y_k denote the last component of y. In the following it will be shown that $y_k = 0$. Then it follows from the proof of Lemma 4.3 that $y = 0^{N+1}$. It follows as in the proof of Lemma 4.3 that $y_{(3,i)} = 0$, $\forall i \in I_m$. Now five different cases are considered.

1. Let $k \in I^-(r) \cup I^+(r)$. This case goes along the same lines as the proof of Lemma 4.3.

2. Let $k = (i', j', -)$ for some $(i', j') \in I^0(r)$. Then $r_{j'}^i \in \{-1, 0\}$, $\forall i \in I_m$. Consequently,

$$0 \;=\; y^\top \partial_{\omega_{j'}^{i'}} \psi_k^{\nu,r,s}(\widetilde{\xi}) = -y_k - y_{(6,j')}\mathcal{I}_{I^0(s)}(j'), \tag{33}$$

$$0 \;=\; y^\top \partial_{\omega_{j'}^{i}} \psi_k^{\nu,r,s}(\widetilde{\xi}) = -y_{(4,i,j')}\mathcal{I}_{I^-(r)}((i,j')) - y_{(6,j')}\mathcal{I}_{I^0(s)}(j'),\ \forall i \in I_m \setminus \{i'\}. \tag{34}$$

There are three possibilities. Either $j' \notin I^0(s)$, implying $y_k = 0$ by (33). Or $j' \in I^0(s)$ and $\exists i'' \in I_m \setminus \{i'\}$ such that $(i'', j') \notin I^-(r)$, implying that $y_{(6,j')} = 0$ by (34), and consequently that $y_k = 0$ by (33). Or $j' \in I^0(s)$ and $(i, j') \in I^-(r)$, $\forall i \in I_m \setminus \{i'\}$. Then $0 = y^\top \partial_{q_{j'}} \psi_k^{\nu,r,s}(\widetilde{\xi}) = -\sum_{i \in I_m \setminus \{i'\}} y_{(4,i,j')} \partial_{q_{j'}} \widehat{l}_{j'}^{\nu,i}(\widetilde{q}) - y_k \partial_{q_{j'}} \widehat{l}_{j'}^{\nu,i'}(\widetilde{q}) = -y_k \sum_{i \in I_m} \partial_{q_{j'}} \widehat{l}_{j'}^{\nu,i}(\widetilde{q})$, where the last equality follows from (33) and (34), implying that $y_k = 0$. The case where $k = (i', j', +)$ for some $(i', j') \in I^0(r)$ goes along the same lines.

3. Let $k \in J^-(r,s)$. It follows that $k \in I^-(s)$, and therefore $v_k > 0$. Clearly,

$$0 = y^\top \partial_{\omega_k^i} \psi_k^{\nu,r,s}(\widetilde{\xi}) = -y_{(4,i,k)}\mathcal{I}_{I^-(r)}((i,k)) + y_{(5,i,k)}\mathcal{I}_{I^+(r)}((i,k)) + y_k,\ \forall i \in I_m. \tag{35}$$

Now, $r_k^i \in \{-1, 0\}$, $\forall i \in I_m$, or $r_k^i \in \{0, +1\}$, $\forall i \in I_m$. In the first case, either $\exists i' \in I_m$, $r_k^{i'} = 0$, and it follows from (35) for $i = i'$ that $y_k = 0$, or $r_k^i = -1$, $\forall i \in I_m$, and (35) yields $y_k = y_{(4,i,k)}$, $\forall i \in I_m$. It follows as in the proof of Lemma 4.3 that $y_{(7,j)} = 0$, $\forall j \in I^-(s) \setminus \{k\}$, and $y_{(8,j)} = 0$, $\forall j \in I^+(s)$. Hence, $0 = y^\top \partial_\mu \psi_k^{\nu,r,s}(\widetilde{\xi}) = -v_k y_{(7,k)}$, implying that $y_{(7,k)} = 0$. Hence, $0 = y^\top \partial_{q_k} \psi_k^{\nu,r,s}(\widetilde{\xi}) = -\sum_{i \in I_m} y_{(4,i,k)} \partial_{q_k} \widehat{l}_k^{\nu,i}(\widetilde{q}) + y_{(7,k)} = -y_k \sum_{i \in I_m} \partial_{q_k} \widehat{l}_k^{\nu,i}(\widetilde{q})$, so it holds that $y_k = 0$. Now consider the case with $r_k^i \in \{0, +1\}$, $\forall i \in I_m$. Since $(r,s) \in \mathcal{T}$ and $s_k = -1$, $\exists i' \in I_m$, $r_k^{i'} \neq +1$. From (35) for $i = i'$ it follows that $y_k = 0$. The case where $k \in J^+(r,s)$ goes along the same lines.

4. Let $k = (j', -)$ for some $j' \in K^-(r,s)$. Either $\exists i' \in I_m$, $(i', j') \notin I^-(r)$, implying $y_{(4,i,j')} = 0$, $\forall (i,j') \in I^-(r)$, and $y_{(5,i,j')} = 0$, $\forall (i,j') \in I^+(r)$, as in the proof of Lemma 4.3. Then $0 = y^\top \partial_{q_{j'}} \psi_k^{\nu,r,s}(\widetilde{\xi}) = y_k$. Or $(i, j') \in I^-(r)$, $\forall i \in I_m$, implying $v_{j'} = 0$, since

$j' \in I^0(s)$ and $(r,s) \in \mathcal{T}$, a contradiction with $j' \in K^-(r,s)$. The case where $k = (j',+)$ for some $j' \in K^+(r,s)$ goes along the same lines.

5. Let $k = 0$. In Lemma 4.3 it was shown that $y_{(7,j)} = 0$, $\forall j \in I^-(s)$, and $y_{(8,j)} = 0$, $\forall j \in I^+(s)$, without using the partial derivatives with respect to μ. Hence, this proof can be used again, and it follows that $0 = y^\top \partial_\mu \psi_k^{\nu,r,s}(\tilde{\xi}) = -y_k$.

Now it has been shown that $\psi_k^{\nu,r,s} \pitchfork \{0^{N+1}\}$, $\forall k \in K$. By the transversality theorem and the continuous differentiability of $\psi_k^{\nu,r,s}$, $\forall k \in K$, it follows that the complement of the set $\{\omega \in \widehat{\Omega}(\nu) \mid \psi_k^{\nu,r,s,\omega} \pitchfork \{0^{N+1}\}\}$ has Lebesgue measure zero. Since $\psi_k^{\nu,r,s,\omega}$ maps from a manifold with dimension $N+1$ into a manifold with dimension $N+1$, and since $\psi_k^{\nu,r,s,\omega}$ is a twice continuously differentiable function, $\psi_k^{\nu,r,s,\omega} \pitchfork \{0^{N+1}\}$ implies that $\psi_k^{\nu,r,s,\omega^{-1}}(\{0^{N+1}\})$ is a 0-dimensional manifold. Q.E.D.

Let some $\nu \in \mathbb{N}$ and some $(r,s) \in \mathcal{T}$ be given. Define $K^2 = \{(k^1,k^2) \in K \times K \mid k^1 \neq k^2, k^1 = (i',j',-) \Rightarrow k^2 \neq (i'',j',+)$, and $k^1 = (i',j',+) \Rightarrow k^2 \neq (i'',j',-)\}$. For $(k^1,k^2) \in K^2$ define a function $\psi_{k^1,k^2}^{\nu,r,s} : \mathbb{R}_{++}^{m(n+1)} \times \mathbb{R}^m \times \widehat{\Omega}(\nu) \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{N+2}$, where $\psi_{k^1,k^2}^{\nu,r,s}$ is the left-hand side of (4)-(11) and the two inequalities corresponding to k^1 and k^2. Define, for every $\omega \in \widehat{\Omega}(\nu)$, the function $\psi_{k^1,k^2}^{\nu,r,s,\omega}$ in the obvious way.

Lemma 4.5

Let $(\{X^i,u^i\}_{i=1}^m, (\hat{l},\hat{L}), p)$ satisfying Assumptions B1-B4 and $v \in Q^n$ be given. Moreover, let $\nu \in \mathbb{N}$, $(r,s) \in \mathcal{T}$, and $(k^1,k^2) \in K^2$ be given. Then $\psi_{k^1,k^2}^{\nu,r,s,\omega} \pitchfork \{0^{N+2}\}$ and $\psi_{k^1,k^2}^{\nu,r,s,\omega^{-1}}(\{0^{N+2}\}) = \emptyset$, except for a set of initial endowments $\omega \in \widehat{\Omega}(\nu)$ with Lebesgue measure zero.

Proof

First it is shown that $\psi_{k^1,k^2}^{\nu,r,s} \pitchfork \{0^{N+2}\}$. Let $\tilde{\xi}$ be such that $\psi_{k^1,k^2}^{\nu,r,s}(\tilde{\xi}) = 0^{N+2}$ and let $y \in \mathbb{R}^{N+2}$ be such that $y^\top M_{k^1,k^2} = 0^{N'^\top}$, where M_{k^1,k^2} denotes the $(N+2) \times N'$ matrix of partial derivatives of $\psi_{k^1,k^2}^{\nu,r,s}$ evaluated at $\tilde{\xi}$. It has to be shown that $y = 0^{N+2}$. Let y_{k^1} and y_{k^2} denote the second last and the last component of y corresponding with the inequalities k^1 and k^2, respectively. It will be shown that $y_{k^1} = 0$ or $y_{k^2} = 0$. Then the proof can be completed as in Lemma 4.4. As before it follows that $y_{(3,i)} = 0$, $\forall i \in I_m$. Ten different cases are considered.

1. Let $k^1 \in I^-(r) \cup I^+(r)$ and $k^2 \in K$. This case follows as Cases 1-5 in the proof of Lemma 4.4.

2. Let $k^1 = (i^1,j^1,-)$ and $k^2 = (i^2,j^2,-)$ for some different $(i^1,j^1),(i^2,j^2) \in I^0(r)$. The case where $j^1 \neq j^2$ follows as in Case 2 of Lemma 4.4. So let $j^1 = j^2$, and therefore $i^1 \neq i^2$. By the definition of K, $r_{j^1}^i \in \{-1,0\}$, $\forall i \in I_m$. It holds that

$$0 = y^\top \partial_{\omega_{j^1}^i} \psi_{k^1,k^2}^{\nu,r,s}(\tilde{\xi}) = -y_{(4,i,j^1)} \mathcal{I}_{I^-(r)}((i,j^1)) - y_{(6,j^1)} \mathcal{I}_{I^0(s)}(j^1), \ \forall i \in I_m \setminus \{i^1,i^2\}, \quad (36)$$

$$0 = y^\top \partial_{\omega_{j^1}^{i^1}} \psi_{k^1,k^2}^{\nu,r,s}(\tilde{\xi}) = -y_{k^1} - y_{(6,j^1)} \mathcal{I}_{I^0(s)}(j^1), \quad (37)$$

$$0 = y^\top \partial_{\omega_{j^1}^{i^2}} \psi_{k^1,k^2}^{\nu,r,s}(\tilde{\xi}) = -y_{k^2} - y_{(6,j^1)} \mathcal{I}_{I^0(s)}(j^1). \quad (38)$$

When $j^1 \notin I^0(s)$, then it follows by (38) that $y_{k^2} = 0$. When $j^1 \in I^0(s)$ and $\exists i^3 \in I_m \setminus \{i^1, i^2\}$ such that $(i^3, j^1) \notin I^-(r)$, then $y_{(6,j^1)} = 0$ by (36). Hence, $y_{k^2} = 0$ by (38). When $j^1 \in I^0(s)$, and $(i, j^1) \in I^-(r)$, $\forall i \in I_m \setminus \{i^1, i^2\}$, then (36), (37), and (38) imply $y_{k^1} = y_{k^2} = y_{(4,i,j^1)}$, $\forall i \in I_m \setminus \{i^1, i^2\}$. Consequently, $0 = y^\top \partial_{q_{j^1}} \psi^{\nu,r,s}_{k^1,k^2}(\widetilde{\xi}) = -\sum_{i \in I_m \setminus \{i^1,i^2\}} y_{(4,i,j^1)} \partial_{q_{j^1}} \widehat{l}^{\nu,i}_{j^1}(\widetilde{q}) - y_{k^1} \partial_{q_{j^1}} \widehat{l}^{\nu,i^1}_{j^1}(\widetilde{q}) - y_{k^2} \partial_{q_{j^1}} \widehat{l}^{\nu,i^2}_{j^1}(\widetilde{q}) = -y_{k^2} \sum_{i \in I_m} \partial_{q_{j^1}} \widehat{l}^{\nu,i}_{j^1}(\widetilde{q})$. Hence $y_{k^2} = 0$. The case $k^1 = (i^1, j^1, +)$ and $k^2 = (i^2, j^2, +)$ is similar. In the case $k^1 = (i^1, j^1, -)$ and $k^2 = (i^2, j^2, +)$ it follows by the definition of K^2 that $j^1 \neq j^2$, and the proof follows as in Case 2 of Lemma 4.4.

3. Let $k^1 = (i^1, j^1, -)$ for some $(i^1, j^1) \in I^0(r)$ and let $k^2 \in J^-(r, s)$. The case where $j^1 \neq k^2$ follows as in Case 2 of Lemma 4.4. So, let $j^1 = k^2$. Since $j^1 \in I^-(s)$ it holds that $v_{j^1} > 0$. By the definition of K, $r^i_{j^1} \in \{-1, 0\}$, $\forall i \in I_m$. Hence,

$$\begin{aligned} 0 = y^\top \partial_{\omega^i_{j^1}} \psi^{\nu,r,s}_{k^1,k^2}(\widetilde{\xi}) &= -y_{(4,i,j^1)} \mathcal{I}_{I^-(r)}((i, j^1)) + y_{k^2},\ \forall i \in I_m \setminus \{i^1\}, && (39) \\ 0 = y^\top \partial_{\omega^{i^1}_{j^1}} \psi^{\nu,r,s}_{k^1,k^2}(\widetilde{\xi}) &= -y_{k^1} + y_{k^2}. && (40) \end{aligned}$$

The case where $\exists i^2 \in I_m \setminus \{i^1\}$ such that $r^{i^2}_{j^1} = 0$ is trivial. Consider the case where $(i, j^1) \in I^-(r)$, $\forall i \in I_m \setminus \{i^1\}$. Then $y_{k^1} = y_{k^2} = y_{(4,i,j^1)}$, $\forall i \in I_m \setminus \{i^1\}$. As in Lemma 4.3 it can be shown that $y_{(7,j)} = 0$, $\forall j \in I^-(s) \setminus \{j^1\}$, and $y_{(8,j)} = 0$, $\forall j \in I^+(s)$. So, $0 = y^\top \partial_\mu \psi^{\nu,r,s}_{k^1,k^2}(\widetilde{\xi}) = -v_{j^1} y_{(7,j^1)}$, implying that $y_{(7,j^1)} = 0$. Hence, $0 = y^\top \partial_{q_{j^1}} \psi^{\nu,r,s}_{k^1,k^2}(\widetilde{\xi}) = -\sum_{i \in I_m \setminus \{i^1\}} y_{(4,i,j^1)} \partial_{q_{j^1}} \widehat{l}^{\nu,i}_{j^1}(\widetilde{q}) - y_{k^1} \partial_{q_{j^1}} \widehat{l}^{\nu,i^1}_{j^1}(\widetilde{q})$, implying that $y_{k^1} = 0$. The cases $k^1 = (i^1, j^1, -)$ and $k^2 \in J^+(r, s)$, $k^1 = (i^1, j^1, +)$ and $k^2 \in J^-(r, s) \cup J^+(r, s)$ are similar.

4. Let $k^1 = (i^1, j^1, -)$ for some $(i^1, j^1) \in I^0(r)$ and let $k^2 = (j^2, -)$ for some $j^2 \in K^-(r, s)$, hence $j^2 \in I^0(s)$ and $v_{j^2} > 0$. The case where $j^1 \neq j^2$ follows as in Case 2 of Lemma 4.4. So, let $j^1 = j^2$. Since $k^1 \in K$ it holds that $r^i_{j^1} \in \{-1, 0\}$, $\forall i \in I_m$. Clearly,

$$\begin{aligned} 0 = y^\top \partial_{\omega^i_{j^1}} \psi^{\nu,r,s}_{k^1,k^2}(\widetilde{\xi}) &= -y_{(4,i,j^1)} \mathcal{I}_{I^-(r)}((i, j^1)) - y_{(6,j^1)},\ \forall i \in I_m \setminus \{i^1\}, \\ 0 = y^\top \partial_{\omega^{i^1}_{j^1}} \psi^{\nu,r,s}_{k^1,k^2}(\widetilde{\xi}) &= -y_{(6,j^1)} - y_{k^1}. \end{aligned}$$

The case where $r^{i^2}_{j^1} = 0$ for some $i^2 \in I_m \setminus \{i^1\}$ is trivial, so consider the case $r^i_{j^1} = -1$, $\forall i \in I_m \setminus \{i^1\}$. Then $y_{(6,j^1)} = -y_{k^1} = -y_{(4,i,j^1)}$, $\forall i \in I_m \setminus \{i^1\}$. As in the proof of Lemma 4.3 it can be shown that $y_{(7,j)} = 0$, $\forall j \in I^-(s)$, and $y_{(8,j)} = 0$, $\forall j \in I^+(s)$. Hence $0 = y^\top \partial_\mu \psi^{\nu,r,s}_{k^1,k^2}(\widetilde{\xi}) = -v_{j^1} y_{k^2}$, implying that $y_{k^2} = 0$. The proof of the cases $k^1 = (i^1, j^1, -)$ and $k^2 = (j^2, +)$, $k^1 = (i^1, j^1, +)$ and $k^2 = (j^2, -)$ or $k^2 = (j^2, +)$, is similar.

5. Let $k^1 = (i^1, j^1, -)$ or $k^1 = (i^1, j^1, +)$ for some $(i^1, j^1) \in I^0(r)$ and let $k^2 = 0$. It can be shown as in Case 2 of Lemma 4.4 that $y_{k^1} = 0$.

6. Let $k^1 \in J^-(r, s) \cup J^+(r, s)$ and let $k^2 \in J^-(r, s) \cup J^+(r, s)$. By definition of the sets $J^-(r, s)$ and $J^+(r, s)$, $\exists i^1 \in I_m$ such that $r^{i^1}_{j^1} = 0$ and then it follows easily that $y_{k^1} = 0$, or $\exists i^2 \in I_m$ such that $r^{i^2}_{j^2} = 0$ and then it follows easily that $y_{k^2} = 0$.

7. Let $k^1 \in J^-(r,s) \cup J^+(r,s)$ and $k^2 = (j^2, -)$ for some $j^2 \in K^-(r,s)$. So $k^1 \in I^-(s) \cup I^+(s)$ and $j^2 \in I^0(s)$, and hence $k^1 \neq j^2$. Therefore it can be shown that $y_{k^2} = 0$ as in Case 4 of Lemma 4.4. The proof for the case where $k^1 \in J^-(r,s) \cup J^+(r,s)$ and $k^2 = (j^2, +)$ for some $j^2 \in K^+(r,s)$ is similar.
8. Let $k^1 \in J^-(r,s)$ and let $k^2 = 0$. Since $k^1 \in I^-(s)$ it holds that $v_{k^1} > 0$ and since $k^2 = 0$ it holds that $\widetilde{\mu} = 1$. In the case $\exists i^1 \in I_m,\ r^{i^1}_{k^1} = 0$, it can be shown as in Case 3 of Lemma 4.4 that $y_{k^1} = 0$. Otherwise, $r^i_{k^1} = -1,\ \forall i \in I_m$. Then,

$$0 = \sum_{i=1}^{m} (\widetilde{x}^i_{k^1} - \omega^i_{k^1}) = \sum_{i=1}^{m} \widehat{l}^{\nu,i}_{k^1}(\widetilde{q}) < 0,$$

since $\widetilde{q}_{k^1} = \widetilde{\mu} v_{k^1} = v_{k^1} > 0$, a contradiction. The case $k^1 \in J^+(r,s)$ and $k^2 = 0$ is similar.
9. Let $k^1 = (j^1, -)$ for some $j^1 \in K^-(r,s)$ or $k^1 = (j^1, +)$ for some $j^1 \in K^+(r,s)$, and $k^2 = (j^2, -)$ for some $j^2 \in K^-(r,s)$ or $k^2 = (j^2, +)$ for some $j^2 \in K^+(r,s)$. In the case $j^1 \neq j^2$ the proof is as in Case 4 of Lemma 4.4. Otherwise, without loss of generality, $k^1 = (j^1, -)$ and $k^2 = (j^1, +)$ for some $j^1 \in I^0(s)$, $0 < v_{j^1} < 1$, and $\exists i^1 \in I_m,\ (i^1, j^1) \notin I^+(r),\ \exists i^2 \in I_m,\ (i^2, j^1) \notin I^-(r)$. It follows easily that $y_{(4,i,j^1)} = 0,\ \forall (i, j^1) \in I^-(r),\ y_{(5,i,j^1)} = 0,\ \forall (i, j^1) \in I^+(r),\ y_{(7,j)} = 0,\ \forall j \in I^-(s)$, and $y_{(8,j)} = 0,\ \forall j \in I^+(s)$. So, $0 = y^\top \partial_{q_{j^1}} \psi^{\nu,r,s}_{k^1,k^2}(\widetilde{\xi}) = y_{k^1} - y_{k^2}$. Moreover, $0 = y^\top \partial_\mu \psi^{\nu,r,s}_{k^1,k^2}(\widetilde{\xi}) = -v_{j^1} y_{k^1} - (1 - v_{j^1}) y_{k^2} = -y_{k^1}$. Hence, $y_{k^1} = y_{k^2} = 0$.
10. Let $k^1 = (j^1, -)$ for some $j^1 \in K^-(r,s)$ or $k^1 = (j^1, +)$ for some $j^1 \in K^+(r,s)$, and let $k^2 = 0$. It can easily be shown that $y_{k^1} = 0$ as in Case 4 of Lemma 4.4.
Now it has been shown that $\psi^{\nu,r,s}_{k^1,k^2} \pitchfork \{0^{N+2}\}$. By the transversality theorem and the twice continuous differentiability of $\psi^{\nu,r,s}_{k^1,k^2}$, it follows that the complement of the set $\{\omega \in \widehat{\Omega}(\nu) \mid \psi^{\nu,r,s,\omega}_{k^1,k^2} \pitchfork \{0^{N+2}\}\}$ has Lebesgue measure zero. Since $\psi^{\nu,r,s,\omega}_{k^1,k^2}$ maps from a manifold with dimension $N+1$ into a manifold with dimension $N+2$, and since $\psi^{\nu,r,s,\omega}_{k^1,k^2}$ is a twice continuously differentiable function, $\psi^{\nu,r,s,\omega}_{k^1,k^2} \pitchfork \{0^{N+2}\}$ implies that ${\psi^{\nu,r,s,\omega}_{k^1,k^2}}^{-1}(\{0^{N+2}\}) = \emptyset$. Q.E.D.

Let $\nu \in \mathbb{N}$, $(r,s) \in \mathcal{T}$, and $\omega \in \Omega(\nu)$ be given. Denote an element $(x, \lambda, q, \mu) \in \mathbb{R}^{m(n+1)}_{++} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ by χ. The function $g^{r,s,\omega} : \mathbb{R}^{m(n+1)}_{++} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^N$ is defined such that $g^{r,s,\omega}(\chi)$ is the left-hand side of (4)-(11). The function $h^{r,s,\omega} : \mathbb{R}^{m(n+1)}_{++} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{N''}$ is defined such that $h^{r,s,\omega}(\chi)$ is the left-hand side of (12)-(20), where N'' denotes the number of inequalities. Define $J^{0,r,s,\omega} = \{j \in I_{N''} \mid h^{r,s,\omega}_j(\chi) = 0\}$, and let $b^{r,s,\omega}(\chi)$ denote the cardinality of this set. The set $D_\omega(r,s)$ is defined by $D_\omega(r,s) = \{\chi \in \mathbb{R}^{m(n+1)}_{++} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \mid g^{r,s,\omega}(\chi) = 0^N$ and $h^{r,s,\omega}(\chi) \geq 0^{N''}\}$. Notice from Theorem 4.2 that

$$C_\omega(r,s) = \{q \in Q^n \mid (x, \lambda, q, \mu) \in D_\omega(r,s) \text{ for some } (x, \lambda, \mu) \in \mathbb{R}^{m(n+1)}_{++} \times \mathbb{R}^m \times \mathbb{R}\},$$

hence $C_\omega(r,s)$ is the image of a projection of $D_\omega(r,s)$. Moreover, it follows by Theorem 4.2 that $f_{|C_\omega(r,s)} : C_\omega(r,s) \to \mathbb{R}^{N+1}$ maps $C_\omega(r,s)$ bijectively onto $D_\omega(r,s)$.

Let T^1 and T^2 be arbitrary subsets of $\mathbb{R}^{N^1}$ and $\mathbb{R}^{N^2}$, respectively. A function $f : T^1 \to T^2$ is of the class C^r if for every $t \in T^1$ there exists an open set U containing t and a C^r map $\tilde{f} : U \to T^2$ such that $f(u) = \tilde{f}(u)$, $\forall u \in T^1 \cap U$. Corollary 3.1.3 in Jongen, Jonker, and Twilt (1983) states that if T^1 is a k-dimensional C^r manifold with boundary and f is a C^r diffeomorphism from T^1 onto T^2, then T^2 is a k-dimensional C^r manifold with boundary and the boundary of T^2 is the image by f of the boundary of T^1. Clearly, $f^{-1}_{|C_\omega(r,s)} : D_\omega(r,s) \to C_\omega(r,s)$ is of the class C^∞. Since, for every $q \in C_\omega(r,s)$, the last component of $f_{|C_\omega(r,s)}(q)$ equals $\frac{q_j}{v_j}$ if $j \in I^-(s)$, and equals $\frac{1-q_j}{1-v_j}$ if $j \in I^+(s)$, it follows that $f_{|C_\omega(r,s)}$ is of the class C^2. In Theorem 4.6 it will be shown, under conditions being satisfied for almost every $\omega \in \Omega$, that $D_\omega(r,s)$ is a 1-dimensional C^2 manifold with boundary. Using the above mentioned result of Jongen, Jonker, and Twilt (1983), it follows then immediately that for almost every $\omega \in \Omega$ the set $C_\omega(r,s)$ is a 1-dimensional C^2 manifold with boundary.

Theorem 4.6
Let $(\{X^i, u^i\}_{i=1}^m, (\hat{l}, \hat{L}), p)$ satisfying Assumptions B1-B4 and $v \in Q^n$ be given. Let $(r,s) \in \mathcal{T}$ and $\omega \in \Omega(\nu)$ be given. Assume that $\psi^{\nu,r,s,\omega} \pitchfork \{0^N\}$, for every $k \in K$, $\psi_k^{\nu,r,s,\omega} \pitchfork \{0^{N+1}\}$, and for every $(k^1,k^2) \in K^2$, $\psi_{k^1,k^2}^{\nu,r,s,\omega} \pitchfork \{0^{N+2}\}$. Then $C_\omega(r,s)$ is a compact 1-dimensional C^2 manifold with boundary.

Proof
It holds that $D_\omega(r,s)$ is a 1-dimensional C^2 MGB if it is a C^2 RCS. Let an element $\tilde{\chi} \in D_\omega(r,s)$ be given. It has to be shown that $\{\partial_\chi g_j^{r,s,\omega}(\tilde{\chi})^\top \mid j \in I_N\} \cup \{\partial_\chi h_j^{r,s,\omega}(\tilde{\chi})^\top \mid j \in J^{0,r,s,\omega}(\tilde{\chi})\}$ is a set of independent vectors. Since, for every $(k^1,k^2) \in K^2$, $\psi_{k^1,k^2}^{\nu,r,s,\omega} \pitchfork \{0^{N+2}\}$, and the domain of $\psi_{k^1,k^2}^{\nu,r,s,\omega}$, $\mathbb{R}_{++}^{m(n+1)} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$, is an $(N+1)$-dimensional manifold, it holds by the definition of transversality that ${\psi_{k^1,k^2}^{\nu,r,s,\omega}}^{-1}(\{0^{N+2}\}) = \emptyset$. Suppose $h_{j^1}^{r,s,\omega}(\tilde{\chi}) = h_{j^2}^{r,s,\omega}(\tilde{\chi}) = 0$ for some $j^1, j^2 \in J^{0,r,s,\omega}(\tilde{\chi})$ with $j^1 \neq j^2$. Since $(\hat{l}, \hat{L})$ is frictionless, it holds that (j^1, j^2) corresponds to some $(k^1,k^2) \in K^2$. Hence $\psi_{k^1,k^2}^{\nu,r,s,\omega}(\tilde{\chi}) = 0^{N+2}$, a contradiction. Consequently, $b^{r,s,\omega}(\tilde{\chi}) \leq 1$. Either $b^{r,s,\omega}(\tilde{\chi}) = 0$, i.e., $J^{0,r,s,\omega}(\tilde{\chi}) = \emptyset$, and $\psi^{\nu,r,s,\omega} \pitchfork \{0^N\}$ yields that $\{\partial_\chi g_j^{r,s,\omega}(\tilde{\chi})^\top \mid j \in I_N\}$ is an independent set of vectors. Or, $b^{r,s,\omega}(\tilde{\chi}) = 1$, and, since $(\hat{l}, \hat{L})$ is frictionless, $J^{0,r,s,\omega}(\tilde{\chi})$ corresponds to an element $k' \in K$. Then $\psi_{k'}^{\nu,r,s,\omega} \pitchfork \{0^{N+1}\}$ yields that $\{\partial_\chi g_j^{r,s,\omega}(\tilde{\chi})^\top \mid j \in I_N\} \cup \{\partial_\chi h_{k'}^{r,s,\omega}(\tilde{\chi})^\top\}$ is an independent set of vectors. Hence, since $g^{r,s,\omega}$ maps from an $(N+1)$-dimensional manifold into $\mathbb{R}^N$, it holds that $D_\omega(r,s)$ is a 1-dimensional C^2 manifold with boundary. Consequently $C_\omega(r,s)$ is a 1-dimensional C^2 manifold with boundary. The compactness of $C_\omega(r,s)$ is easily shown. Q.E.D.

Notice that almost every $\omega \in \Omega(\nu)$ satisfies the finite number of requirements in Theorem 4.6, by Lemmas 4.3, 4.4, and 4.5, and by the fact that $\Omega(\nu) \subset \hat{\Omega}(\nu)$. Since there is a countable number of sets $\Omega(\nu)$, Theorem 4.6 holds for almost every $\omega \in \Omega$. In Definition 4.7, initial endowments $\omega \in \Omega$ are defined to be regular, if the set C_ω has a nice structure.

Definition 4.7

Let $(\{X^i, u^i\}_{i=1}^m, (\widehat{l}, \widehat{L}), p)$ satisfying Assumptions B1-B4 and $v \in Q^n$ be given. The set of regular initial endowments, denoted by Ω^*, is the set of initial endowments $\omega \in \Omega$ for which the components of the set C_ω are given by (1) the point v as an isolated point being a Drèze equilibrium, or an arc containing v and precisely one Drèze equilibrium both being boundary points of the arc, (2) a finite number of arcs containing precisely two Drèze equilibria both being boundary points, and (3) a finite number of loops containing neither v nor any Drèze equilibrium.

Let some $\omega \in \Omega^*$ be given. Then the quantity adjustment process satisfies the convergence criterion given in Definition 3.3. Moreover, it follows immediately that the number of Drèze equilibria is odd. In Theorem 4.8 conditions are given under which $\omega \in \Omega$ is regular. Clearly, these conditions are satisfied for almost every $\omega \in \Omega$. In fact, under these conditions it follows from the proof of Theorem 4.8 that every component being an arc or a loop is a 1-dimensional piecewise twice continuously differentiable manifold.

The proof follows the interpretation of the quantity adjustment process given in Section 3. Let some $(r, s) \in \mathcal{T}$ be given. By Theorem 4.6 it holds that $C_\omega(r, s)$ is a compact 1-dimensional C^2 manifold with boundary, and therefore a finite union of finite disjoint sets each being diffeomorphic to either the unit circle or the closed unit interval. Denote these different sets by $C_\omega(r, s, 1), \ldots, C_\omega(r, s, k(r, s))$. Let $C_\omega(r, s, k)$ be given, for some $k \in I_{k(r,s)}$. In the proof of Theorem 4.8 it will be shown that if q is a boundary point of $C_\omega(r, s, k)$, then either there is a unique sign vector $(r', s') \in \mathcal{T}$ such that $q \in C_\omega(r', s', k')$ for some $k' \in I_{k(r',s')}$ and q is a boundary point of this set, or q is a Drèze equilibrium, or q equals the initial state v. In the first case, either $r' = r$ and s' differs from s in only one component, or $s' = s$ and r' differs from r in only one component. Hence, at most one market attains an equilibrium at the same time, unless a Drèze equilibrium is reached by the process.

Theorem 4.8

Let $(\{X^i, u^i\}_{i=1}^m, (\widehat{l}, \widehat{L}), p)$ satisfying Assumptions B1-B4 and $v \in Q^n$ be given. Let $\nu \in \mathbb{N}$ and $\omega \in \Omega(\nu)$ be given. Assume that, for every $(r, s) \in \mathcal{T}$, $\psi^{\nu,r,s,\omega} \pitchfork \{0^N\}$, for every $k \in K$, $\psi_k^{\nu,r,s,\omega} \pitchfork \{0^{N+1}\}$, and for every $(k^1, k^2) \in K^2$, $\psi_{k^1,k^2}^{\nu,r,s,\omega} \pitchfork \{0^{N+2}\}$. Then $\omega \in \Omega^*$.

Proof

Consider some $(r, s) \in \mathcal{T}$ and some boundary point $q \in C_\omega(r, s)$. Let $f(q, \omega) = (x, \lambda, q, \mu) = \chi$. So $b^{r,s,\omega}(\chi) = 1$. Hence, there is exactly one $j \in I_{N''}$ such that $h_j^{r,s,\omega}(\chi) = 0$. Clearly, j corresponds to a unique element $k^1 \in K$. It will be shown that q is an element of exactly one set $C_\omega(r', s', k')$ such that $(r, s, k) \neq (r', s', k')$. Five different cases have to be distinguished.

1. $k^1 = (i^1, j^1) \in I^-(r) \cup I^+(r)$. Define $\tilde{r}^{i^1}_{j^1} = 0$, and $\tilde{r}^i_j = r^i_j$, $\forall (i,j) \in I_m \times I_n \setminus \{i^1, j^1\}$. Clearly, $(\tilde{r}, s) \in \mathcal{T}$ and $q \in C_\omega(\tilde{r}, s)$. Notice that $b^{\tilde{r},s,\omega}(\chi) = 1$, so q is a boundary point of $C_\omega(\tilde{r}, s)$.

Let $(\hat{r}, \hat{s}) \in \mathcal{T}$ be such that $q \in C_\omega(\hat{r}, \hat{s})$. It will be shown that $(\hat{r}, \hat{s}) = (r, s)$ or $(\hat{r}, \hat{s}) = (\tilde{r}, s)$. Suppose $\exists j^2 \in I^-(\hat{s}) \cap I^0(s)$, then $q_{j^2} = \mu v_{j^2}$ and $\hat{z}_{j^2}(q, \omega) = 0$. If $v_{j^2} > 0$ and $\exists i' \in I_m$, $r^{i'}_{j^2} \neq +1$, then $(k^1, (j^2, -)) \in K^2$, and $\chi \in \psi^{\nu,r,s,\omega^{-1}}_{k^1,(j^2,-)}(\{0^{N+2}\}) = \emptyset$, a contradiction. Consequently, $v_{j^2} = 0$ or $(i, j^2) \in I^+(r)$, $\forall i \in I_m$. Since $j^2 \in I^-(\hat{s})$ it holds that $v_{j^2} > 0$, so $(i, j^2) \in I^+(r)$, $\forall i \in I_m$. Since $\hat{z}_{j^2}(q, \omega) = 0$, this implies $1 = q_{j^2} = \mu v_{j^2}$, hence $\mu = 1$. Hence $\chi \in \psi^{\nu,r,s,\omega^{-1}}_{k,0}(\{0^{N+2}\}) = \emptyset$, a contradiction.

Suppose $\exists j^2 \in I^-(\hat{s}) \cap I^+(s)$, then $q_{j^2} = \mu v_{j^2}$ and $(1 - q_{j^2}) = \mu(1 - v_{j^2})$ implies $\mu = 1$, so $\chi \in \psi^{\nu,r,s,\omega^{-1}}_{k^1,0}(\{0^{N+2}\}) = \emptyset$, a contradiction. Similarly, it can be shown that $I^+(\hat{s}) \cap (I^-(s) \cup I^0(s)) = \emptyset$.

Suppose $\exists j^2 \in I^0(\hat{s}) \cap I^-(s)$, then $v_{j^2} > 0$ and $\hat{z}_{j^2}(q, \omega) = 0$. Since $\psi^{\nu,r,s,\omega^{-1}}_{k^1,j}(\{0^{N+2}\}) = \emptyset$, $\forall j \in J^-(r, s)$, it holds that $j^2 \notin J^-(r, s)$. So $r^i_{j^2} = -1$, $\forall i \in I_m$. Since $\hat{z}_{j^2}(q, \omega) = 0$, it follows that $q_{j^2} = 0$. Since $j^2 \in I^-(s)$ it holds that $q_{j^2} = \mu v_{j^2}$. Hence $\mu = 0$, and consequently $\hat{z}(q, \omega) = 0^{n+1}$. Since $J^-(\hat{r}, \hat{s}) \cup J^+(\hat{r}, \hat{s}) \neq \emptyset$, a contradiction is obtained with $\psi^{\nu,r,s,\omega^{-1}}_{k^1,j^3}(\{0^{N+2}\}) = \emptyset$, for any $j^3 \in J^-(\hat{r}, \hat{s}) \cup J^+(\hat{r}, \hat{s})$. Similarly, it can be shown that $I^0(\hat{s}) \cap I^+(s) = \emptyset$.

Hence $\hat{s} = s$. In the case $\hat{r} \neq \tilde{r}$ and $\hat{r} \neq r$ at least one of the inequalities in (12)-(15) is binding, giving a contradiction as before.

2. $k^1 = (i^1, j^1, -)$ for some $(i^1, j^1) \in I^0(r)$ or $k^1 = (i^1, j^1, +)$ for some $(i^1, j^1) \in I^0(r)$. Define $\tilde{r}^{i^1}_{j^1} = -1$, if $k^1 = (i^1, j^1, -)$ and $\tilde{r}^{i^1}_{j^1} = +1$ if $k^1 = (i^1, j^1, +)$, and $\tilde{r}^i_j = r^i_j$, $\forall (i,j) \in I_m \times I_n \setminus \{(i^1, j^1)\}$. Consider the case where $k^1 = (i^1, j^1, -)$. Then $(\tilde{r}, s) \in \mathcal{T}$ and $q \in C_\omega(\tilde{r}, s)$, unless $\tilde{r}^i_{j^1} = -1$, $\forall i \in I_m$, and $s_{j^1} = +1$ or both $s_{j^1} = 0$ and $v_{j^1} > 0$. In the first case it follows that $\hat{z}_{j^1}(q, \omega) = 0$, and since $j^1 \in J^+(r, s)$ it holds that $\chi \in \psi^{\nu,r,s,\omega^{-1}}_{k^1,j^1}(\{0^{N+2}\}) = \emptyset$, a contradiction. In the second case it follows that $\hat{z}_{j^1}(q, \omega) = 0$, hence $q_{j^1} = 0$, and therefore $\mu = 0$. So $\hat{z}(q, \omega) = 0^{n+1}$, and since $J^-(r, s) \cup J^+(r, s) \neq \emptyset$, a contradiction is obtained with $\psi^{\nu,r,s,\omega^{-1}}_{k^1,j^2}(\{0^{n+2}\}) = \emptyset$, for any $j^2 \in J^-(r, s) \cup J^+(r, s)$. So $(\tilde{r}, s) \in \mathcal{T}$. Similarly it can be shown that $(\tilde{r}, s) \in \mathcal{T}$, if $k^1 = (i^1, j^1, +)$. Notice that $b^{\tilde{r},s,\omega}(\chi) = 1$, so q is a boundary point of $C_\omega(\tilde{r}, s)$.

Let $(\hat{r}, \hat{s}) \in \mathcal{T}$ be such that $q \in C_\omega(\hat{r}, \hat{s})$. As in Case 1 it can be shown that $(\hat{r}, \hat{s}) = (r, s)$ or $(\hat{r}, \hat{s}) = (\tilde{r}, s)$.

3. $k^1 \in J^-(r, s) \cup J^+(r, s)$. Define $\tilde{s}_{k^1} = 0$ and $\tilde{s}_j = s_j$, $\forall j \in I_n \setminus \{k^1\}$. Then $(r, \tilde{s}) \in \mathcal{T}$, $q \in C_\omega(r, \tilde{s})$, and $b^{r,\tilde{s},\omega}(\chi) = 1$, so q is a boundary point of $C_\omega(r, \tilde{s})$, unless $\tilde{s} = 0^n$, or $v_{k^1} < 1$ and $r^i_{k^1} = +1$, $\forall i \in I_m$, or $v_{k^1} > 0$ and $r^i_{k^1} = -1$, $\forall i \in I_m$. In the first case it is clear that q is a Drèze equilibrium. In the second case it follows that $\hat{z}_{k^1}(q, \omega) = 0$, hence $q_{k^1} = 1$, and therefore $\mu = 0$. So $\hat{z}(q, \omega) = 0^{n+1}$ and q is a Drèze equilibrium. In the third

case it follows that $\widehat{z}_{k^1}(q,\omega)=0$, hence $q_{k^1}=0$, and therefore $\mu=0$. So $\widehat{z}(q,\omega)=0^{n+1}$ and q is a Drèze equilibrium.
Let $(\widehat{r},\widehat{s})\in\mathcal{T}$ be such that $q\in C_\omega(\widehat{r},\widehat{s})$. It follows easily that $\widehat{r}=r$. As in Case 1 it can be shown that $I^-(\widehat{s})\cap(I^0(s)\cup I^+(s))=\emptyset$ and $I^+(\widehat{s})\cap(I^-(s)\cup I^0(s))=\emptyset$. Suppose $\exists j^2\in I^0(\widehat{s})\cap I^-(s)$, then $v_{j^2}>0$ and $\widehat{z}_{j^2}(q,\omega)=0$. Now $j^2\notin J^-(r,s)\backslash\{k^1\}$, since otherwise $\psi_{k^1,j^2}^{\nu,r,s,\omega^{-1}}(\{0^{N+2}\})=\emptyset$ yields a contradiction. So $j^2=k^1$ or $r^i_{j^2}=-1$, $\forall i\in I_m$. However, this latter case is excluded since $(r,\widehat{s})\in\mathcal{T}$, $v_{j^2}>0$, and $\widehat{s}_{j^2}=0$ implies $r^{i'}_{j^2}\neq-1$ for some $i'\in I_m$. When $(r,\widetilde{s})\notin\mathcal{T}$ it follows that $\widetilde{s}=0^n$, since $v_{k^1}<1$ and $r^i_{k^1}=+1$, $\forall i\in I_m$, or $v_{k^1}>0$ and $r^i_{k^1}=-1$, $\forall i\in I_m$, is excluded. However, $\widetilde{s}=0^n$ and $I^0(\widehat{s})\cap I^-(s)=\{k^1\}$ implies $\widehat{s}=0^n$, a contradiction. Similarly, it can be shown that $I^0(\widehat{s})\cap I^+(s)=\emptyset$ in the case $(r,\widetilde{s})\notin\mathcal{T}$, and $I^0(\widehat{s})\cap I^-(s)=\emptyset$ or $I^0(\widehat{s})\cap I^+(s)=\{k^1\}$ in the case $(r,\widetilde{s})\in\mathcal{T}$. Consequently, $(r,\widetilde{s})\notin\mathcal{T}$ implies $\widehat{s}=s$, and $(r,\widetilde{s})\in\mathcal{T}$ implies $\widehat{s}\in\{s,\widetilde{s}\}$.
4. $k^1=(j^1,-)$ for some $j^1\in K^-(r,s)$ or $k^1=(j^1,+)$ for some $j^1\in K^+(r,s)$. Define $\widetilde{s}_{k^1}=-1$ if $k^1=(j^1,-)$ and $\widetilde{s}_{k^1}=+1$ if $k^1=(j^1,+)$, and $\widetilde{s}_j=s_j$, $\forall j\in I_n\setminus\{k^1\}$. Then $(r,\widetilde{s})\in\mathcal{T}$ and $q\in C_\omega(r,\widetilde{s})$. Notice that $b^{r,\tilde{s},\omega}(\chi)=1$, so q is a boundary point of $C_\omega(r,\widetilde{s})$. Let $q\in C_\omega(\widehat{r},\widehat{s})$. It is easily shown that $\widehat{r}=r$. Suppose $\exists j^2\in I^-(\widehat{s})\cap I^0(s)$. Then $j^2\in K^-(r,s)$ and $\chi\in\psi_{k^1,(j^2,-)}^{\nu,r,s,\omega^{-1}}(\{0^{N+2}\})=\emptyset$, yielding a contradiction, unless $v_{j^2}=0$, or $(i,j^2)\in I^+(r)$, $\forall i\in I_m$, or $j^2=j^1$. Since $j^2\in I^-(\widehat{s})$ implies $v_{j^2}>0$ and $\exists i^2\in I_m$, $r^{i^2}_{j^2}\neq+1$, it holds that $j^2=j^1$. Moreover, $k^1=(j^1,-)$, since otherwise $\mu=1$ and $\chi\in\psi_{(j^1,+),0}^{\nu,r,s,\omega^{-1}}(\{0^{N+2}\})=\emptyset$, a contradiction. Similarly it can be shown that $j^2\in I^+(\widehat{s})\cap I^0(s)$ implies $j^1=j^2$ and $k^1=(j^1,+)$. Finally, it can be shown as in Case 1 that $I^-(\widehat{s})\cap I^+(s)=\emptyset$, $I^+(\widehat{s})\cap I^-(s)=\emptyset$, and $I^0(\widehat{s})\cap(I^-(s)\cup I^+(s))=\emptyset$. So $\widehat{s}=s$ or $\widehat{s}=\widetilde{s}$.
5. $k^1=0$. So $\mu=1$ and $q=v$. It will be shown that $q\in C_\omega(\widehat{r},\widehat{s})$ implies $(\widehat{r},\widehat{s})=(r,s)$. It is easily shown that $\widehat{r}=r$. Suppose $\exists j^2\in I^-(\widehat{s})\cap I^0(s)$, then $q_{j^2}=\mu v_{j^2}$ and $\widehat{z}_{j^2}(q,\omega)=0$. If $j^2\in K^-(r,s)$, then $\chi\in\psi_{(j^2,-),0}^{\nu,r,s,\omega^{-1}}(\{0^{N+2}\})=\emptyset$, a contradiction. So $j^2\notin K^-(r,s)$ and hence $v_{j^2}=0$ or $(i,j^2)\in I^+(r)$, $\forall i\in I_m$. Since $j^2\in I^-(\widehat{s})$ implies $v_{j^2}>0$ and $\exists i^2\in I_m$, $r^{i^2}_{j^2}\neq+1$, it holds that $I^-(\widehat{s})\cap I^0(s)=\emptyset$. Suppose $\exists j^2\in I^-(\widehat{s})\cap I^+(s)$, then $\widehat{z}_{j^2}(q,\omega)=0$. Since $j^2\in I^-(\widehat{s})$ implies $\exists i^2\in I_m$, $r^{i^2}_{j^2}\neq+1$, it follows from $j^2\in I^+(s)$ that $j^2\in J^+(r,s)$. This contradicts $\psi_{j^2,0}^{\nu,r,s,\omega^{-1}}(\{0^{N+2}\})=\emptyset$. Similarly it can be shown that $I^+(\widehat{s})\cap(I^-(s)\cup I^0(s))=\emptyset$. Suppose $\exists j^2\in I^0(\widehat{s})\cap I^-(s)$. As in Case 1 it can be shown that $\mu=0$, contradicting $\mu=1$. Similarly it can be shown that $I^0(\widehat{s})\cap I^+(s)=\emptyset$. Consequently $\widehat{s}=s$. It is easily shown that $\widehat{z}(q,\omega)\neq0^{n+1}$, since otherwise $\chi\in\psi_{j^3,0}^{\nu,r,s,\omega^{-1}}(\{0^{N+2}\})=\emptyset$, for any $j^3\in J^-(r,s)\cup J^+(r,s)$, a contradiction.
Using Cases 1-5 it follows that the intersection of two different sets $C_\omega(r^1,s^1,k^1)$ and $C_\omega(r^2,s^2,k^2)$ is either empty or a common boundary point. Moreover, it follows that if q^1 is a boundary point of $C_\omega(r^1,s^1,k^1)$, then either $q^1=v$, or q^1 is a Drèze equilibrium, or q^1 is a boundary point of a uniquely determined set $C_\omega(r^2,s^2,k^2)$ for some $(r^2,s^2)\neq(r^1,s^1)$. In the last case it follows that $C_\omega(r^2,s^2,k^2)$ is an arc and therefore has another boundary

point. For the other boundary point of $C_\omega(r^2, s^2, k^2)$, say q^2, it holds that either $q^2 = v$, or q^2 is a Drèze equilibrium, or q^2 is a boundary point of a uniquely determined set $C_\omega(r^3, s^3, k^3)$. Using the finiteness of the number of sets $C_\omega(r, s, k)$ it follows that in this way after a finite number of, say M, steps a boundary point q^M is reached satisfying, either $q^M = v$, or q^M is a Drèze equilibrium, or $q^M = q^1$. Using the fact that the intersection of three different sets is empty, it is easily seen that the components of $\cup_{(r,s)\in\mathcal{T}} C_\omega(r, s)$ are given by a finite number of loops and a finite number of arcs, where the boundary points of any arc are either v and a Drèze equilibrium or two different Drèze equilibria.
Let v be such that $\widehat{z}(v, \omega) = 0^{n+1}$, and $v_j = 0$ or $v_j = 1$, $\forall j \in I_n$. Let $f(v) = (x, \lambda, v, \mu) = \chi$. For every $j \in I_n$, it will be shown that $v_j = 0$ implies $-\partial_{x^i_j} u^i(x^i) + \lambda^i p_j > 0,\ \forall i \in I_m$, and $v_j = 1$ implies $\partial_{x^i_j} u^i(x^i) - \lambda^i p_j > 0,\ \forall i \in I_m$. Suppose there exists an element $(i^1, j^1) \in I_m \times I_n$ such that $v_{j^1} = 0$ and $-\partial_{x^{i^1}_{j^1}} u^{i^1}(x^{i^1}) + \lambda^{i^1} p_{j^1} \leq 0$. Clearly, this inequality is binding, since $v_{j^1} = 0$ implies that demand rationing cannot be binding on market j^1. Define $s \in \mathcal{S}$ by $s_{j^1} = -1,\ s_j = 0,\ \forall j \in I_n \setminus \{j^1\}$, and define $r \in \mathcal{R}$ by $r^i_j = +1$, if $\partial_{x^i_j} u^i(x^i) - \lambda^i p_j > 0$, $r^i_j = 0$, if $\partial_{x^i_j} u^i(x^i) - \lambda^i p_j = 0$, and $r^i_j = -1$, if $-\partial_{x^i_j} u^i(x^i) + \lambda^i p_j > 0$, $\forall (i, j) \in I_m \times I_n$. It is easily verified that $(r, s) \in \mathcal{T}$ and that $j^1 \in J^-(r, s)$. Since $\chi \in \psi^{\nu,r,s,\omega^{-1}}_{(i^1,j^1,-),j^1}(\{0^{N+2}\}) = \emptyset$, a contradiction is obtained. Similarly, it can be shown that, for every $j \in I_n$, $v_j = 1$ implies $\partial_{x^i_j} u^i(x^i) - \lambda^i p_j > 0,\ \forall i \in I_m$. Suppose $v \in C_\omega(\tilde{r}, \tilde{s})$ for some $(\tilde{r}, \tilde{s}) \in \mathcal{T}$. It is easily shown that for every $j \in I_n$, $\tilde{r}^i_j = -1,\ \forall i \in I_m$, if $v_j = 0$, and $\tilde{r}^i_j = +1,\ \forall i \in I_m$, if $v_j = 1$. Let $j' \in I_n$ be such that $v_{j'} = 0$. Then $(\tilde{r}, \tilde{s}) \in \mathcal{T}$ implies $\tilde{s}_{j'} \geq 0$. Since $\tilde{r}^i_{j'} = -1,\ \forall i \in I_m$, it holds that $\tilde{s}_{j'} = 0$. Similarly, it can be shown that if $v_j = 1$ for some $j \in I_n$, then $\tilde{s}_j = 0$. Consequently $\tilde{s} = 0^n$, a contradiction. Since $C_\omega(r, s)$ is compact for every $(r, s) \in \mathcal{T}$ it follows, using Theorem 4.1, that v is an isolated point of the set C_ω. Q.E.D.

Finally, it will be shown that the closure of the set of non-regular initial endowments has Lebesgue measure zero. A preliminary lemma is needed first. Let $\nu \in \mathbb{N}$ and $(r, s) \in \mathcal{T}$ be given. Define the correspondences $Q^{\nu,r,s} : \widehat{\Omega}(\nu) \to Q^n$ and $Q^\nu : \widehat{\Omega}(\nu) \to Q^n$ by

$$\begin{aligned} Q^{\nu,r,s}(\omega) &= \{q \in Q^n \mid \exists (x, \lambda, q, \mu) \in \mathbb{R}^{N+1} \text{ such that } (x, \lambda, \omega, q, \mu) \text{ satisfies (4)-(20)} \\ &\qquad \text{corresponding to } (r, s) \text{ and } \nu\},\ \forall \omega \in \widehat{\Omega}(\nu), \\ Q^\nu(\omega) &= \cup_{(r,s)\in\mathcal{T}} Q^{\nu,r,s}(\omega) \cup \{v\},\ \forall \omega \in \widehat{\Omega}(\nu). \end{aligned}$$

Notice that $Q^\nu(\omega) = C_\omega,\ \forall \omega \in \Omega(\nu)$, according to Theorems 3.1, 4.1, and 4.2.

Lemma 4.9

Let $(\{X^i, u^i\}_{i=1}^m, (\widehat{l}, \widehat{L}), p)$ satisfying Assumptions B1-B4 and $v \in Q^n$ be given. Moreover, let $\nu \in \mathbb{N}$ be given. Then the correspondence Q^ν is compact-valued and upper semi-continuous.

Proof

Let $(\omega^t)_{t\in\mathbb{N}}$ be a sequence in $\widehat{\Omega}(\nu)$ converging to $\overline{\omega} \in \widehat{\Omega}(\nu)$ and let $(q^t)_{t\in\mathbb{N}}$ be a sequence in

Q^n such that $q^t \in Q^\nu(\omega^t)$. It will be shown that $(q^t)_{t\in\mathbb{N}}$ has a subsequence which converges to a point $\overline{q} \in Q^\nu(\overline{\omega})$. In the case $q^t = v$ for an infinite number of elements in the sequence, Lemma 4.9 is easily shown. Hence consider the opposite case. Since the set of sign vectors $\mathcal{T}$ is finite and the set Q^n is compact, it can be assumed without loss of generality that $\exists(\overline{r},\overline{s}) \in \mathcal{T},\ \forall t \in \mathbb{N},\ q^t \in Q^{\nu,\overline{r},\overline{s}}(\omega^t),\ q^t \to \overline{q} \in Q^n$. Since the left-hand side of (4)-(20) is continuous as a function of (x,λ,ω,q,μ) it follows that $\overline{q} \in Q^{\nu,\overline{r},\overline{s}}(\overline{\omega}) \subset Q^\nu(\overline{\omega})$. Q.E.D.

Theorem 4.10

Let $(\{X^i,u^i\}_{i=1}^m,(\hat{l},\hat{L}),p)$ satisfying Assumptions B1-B4 and $v \in Q^n$ be given. Then, the set of non-regular initial endowments $\Omega\setminus\Omega^*$ has a closure in Ω with Lebesgue measure zero.

Proof

Let $\nu \in \mathbb{N}$ be given. First it is shown that the closure of the set $\Omega(\nu)\setminus\Omega^*$ in the closure of $\Omega(\nu)$ in Ω has Lebesgue measure zero. Denote this set by $\Pi(\nu)$. Define the function $f^\nu : Q^n \times \hat{\Omega}(\nu) \to \mathbb{R}^{N+1}$ by

$$f^\nu(q,\omega) = ((d^i(\hat{l}^{\nu,i}(q),\hat{L}^{\nu,i}(q),\omega))_{i=1}^m,(\partial_{x^i_{n+1}}u^i(d^i(\hat{l}^{\nu,i}(q),\hat{L}^{\nu,i}(q),\omega)))_{i=1}^m,q,\hat{\mu}(q)),\ \forall q \in Q^n,\ \forall \omega \in \hat{\Omega}(\nu).$$

In the case $\omega \in \Omega(\nu)\setminus\Omega^*$ there exists by Theorem 4.8 an element $q \in Q^n$ such that (q,ω) belongs to the set $\Sigma(\nu)$ defined by

$$\begin{aligned}
\Sigma(\nu) \quad = \quad & \{(q,\omega) \in Q^n \times \hat{\Omega}(\nu) \mid q \in Q^\nu(\omega) \text{ and} \\
& \exists(r,s) \in \mathcal{T} \text{ such that } \psi^{\nu,r,s,\omega}(f^\nu(q,\omega)) = 0^N \text{ and } \operatorname{rank}\partial\psi^{\nu,r,s,\omega}(f^\nu(q,\omega)) \le N-1, \text{ or} \\
& \exists(r,s) \in \mathcal{T},\ \exists k \in K \text{ such that } \psi_k^{\nu,r,s,\omega}(f^\nu(q,\omega)) = 0^{N+1} \text{ and } \operatorname{rank}\partial\psi_k^{\nu,r,s,\omega}(f^\nu(q,\omega)) \le N, \text{ or} \\
& \exists(r,s) \in \mathcal{T},\ \exists(k^1,k^2) \in K^2 \text{ such that } \psi_{k^1,k^2}^{\nu,r,s,\omega}(f^\nu(q,\omega)) = 0^{N+2} \text{ and } \operatorname{rank}\partial\psi_{k^1,k^2}^{\nu,r,s,\omega}(f^\nu(q,\omega)) \le N+1\}.
\end{aligned}$$

It is easily shown that $\Sigma(\nu)$ is closed in $Q^n \times \hat{\Omega}(\nu)$, since Σ can be obtained by a finite union of sets being closed in $Q^n \times \hat{\Omega}(\nu)$, due to the continuity of the functions f^ν, $\psi^{\nu,r,s}$, $\psi_k^{\nu,r,s}$, and $\psi_{k^1,k^2}^{\nu,r,s}$, intersected with the set $\{(q,\omega) \in Q^n \times \hat{\Omega}(\nu) \mid q \in Q^\nu(\omega)\}$, which is closed by the upper semi-continuity of Q^ν. Define the projection $\pi : \Sigma(\nu) \to \hat{\Omega}(\nu)$ by $\pi(q,\omega) = \omega,\ \forall(q,\omega) \in \Sigma(\nu)$. Then $\Omega(\nu)\setminus\Omega^* \subset \pi(\Sigma(\nu))$ and $\pi(\Sigma(\nu))$ is a subset of a set with measure zero by Lemmas 4.3, 4.4, and 4.5.

It will be shown that $\pi(\Sigma(\nu))$ is closed in $\hat{\Omega}(\nu)$. Since the image by a continuous proper mapping of a closed set is closed, it is sufficient to show that π is proper. Let W be a compact subset of $\hat{\Omega}(\nu)$. It has to be shown that $\pi^{-1}(W)$ is compact. By the continuity of π it holds that $\pi^{-1}(W)$ is closed in $\Sigma(\nu)$. Since $\Sigma(\nu)$ is closed in $Q^n \times \hat{\Omega}(\nu)$ it follows that $\pi^{-1}(W)$ is closed in $Q^n \times \hat{\Omega}(\nu)$. Obviously, $\pi^{-1}(W)$ is a subset of the set $\{(q,\omega) \in Q^n \times W \mid q \in Q^\nu(W)\}$, being compact by the compact-valuedness and upper semi-continuity of Q^ν (Lemma 4.9) and the compactness of W. Consequently, $\pi^{-1}(W)$ is a closed subset of a compact set and is therefore compact. So the function π is proper and hence $\pi(\Sigma(\nu))$ is closed in $\hat{\Omega}(\nu)$. Notice that $\Pi(\nu)$ is contained in $\pi(\Sigma(\nu))$.

Next it is shown that $\cup_{\nu \in \mathbb{N}} \Pi(\nu)$ is closed in Ω, and since this set contains all non-regular initial endowments, Theorem 4.10 is proved. Let $(\omega^t)_{t \in \mathbb{N}}$ be a convergent sequence in $\cup_{\nu \in \mathbb{N}} \Pi(v)$, and let $\overline{\omega} \in \Omega$ denote its limit. Since $(\widehat{l}, \widehat{L})$ is locally constant, $\exists \overline{\nu} \in \mathbb{N}$ such that $(\omega^t)_{t \in \mathbb{N}}$ has an infinite number of elements in $\Omega(\overline{\nu})$. Clearly, $\overline{\omega} \in \Pi(\overline{\nu}) \subset \cup_{\nu \in \mathbb{N}} \Pi(\nu)$. Q.E.D.

The following corollaries follow immediately.

Corollary 4.11
Let $(\{X^i, u^i\}_{i=1}^m, (\widehat{l}, \widehat{L}), p)$ *satisfying Assumptions B1-B4 and* $v \in Q^n$ *be given. Then the price adjustment process for the economy* $\mathcal{E}_\omega = (\{X^i, u^i, \omega^i\}_{i=1}^m, (\widehat{l}, \widehat{L}), p)$ *with initial state* $v \in Q^n$ *converges, except for a set of initial endowments in* Ω *having a closure in* Ω *with Lebesgue measure zero.*

Corollary 4.12
Let $(\{X^i, u^i\}_{i=1}^m, (\widehat{l}, \widehat{L}), p)$ *satisfying Assumptions B1-B4 be given. Then the number of Drèze equilibria of the economy* $\mathcal{E} = (\{X^i, u^i, \omega^i\}_{i=1}^m, (\widehat{l}, \widehat{L}), p)$ *is a finite and odd number, except for a set of initial endowments in* Ω *having a closure in* Ω *with Lebesgue measure zero.*

Corollary 4.12 extends Theorem 1.6b of Laroque and Polemarchakis (1978), where it is shown that there exists a finite number of Drèze equilibria. Moreover, their assumptions exclude rationing determined by priority, a case not excluded in this paper.

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